how many ways are there to traverse a graph?

ho boon suan

hbs@u.nus.edu

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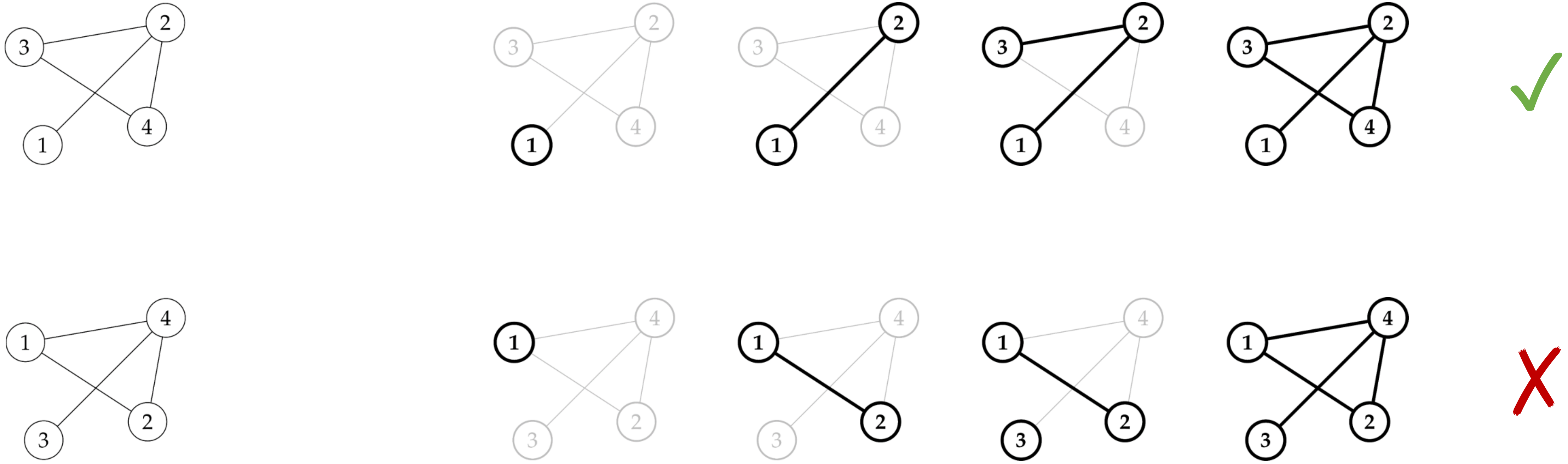
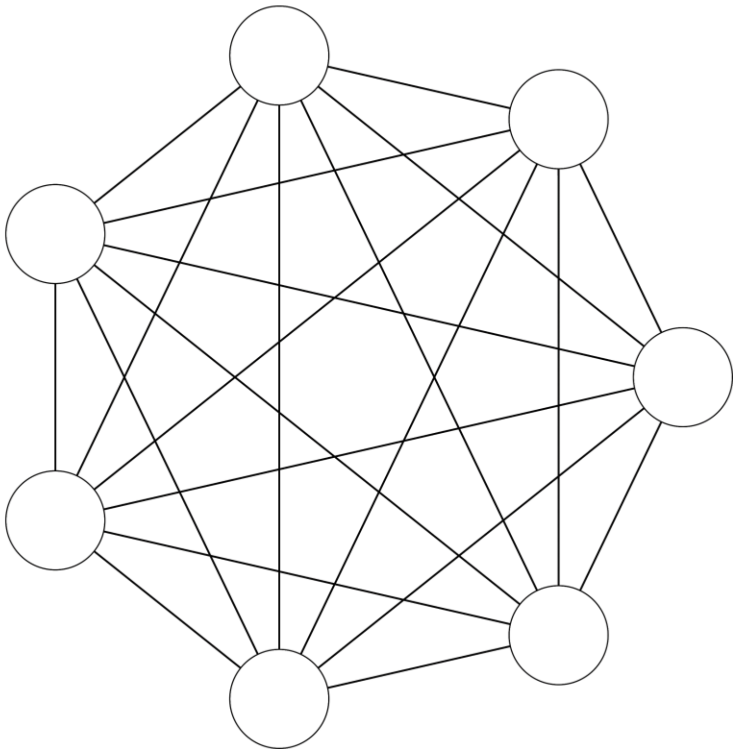
a vertex ordering $v_1, \ldots, v_n$ is *successive* if
for each $j > 1$, there exists $i < j$ with v_i — v_j

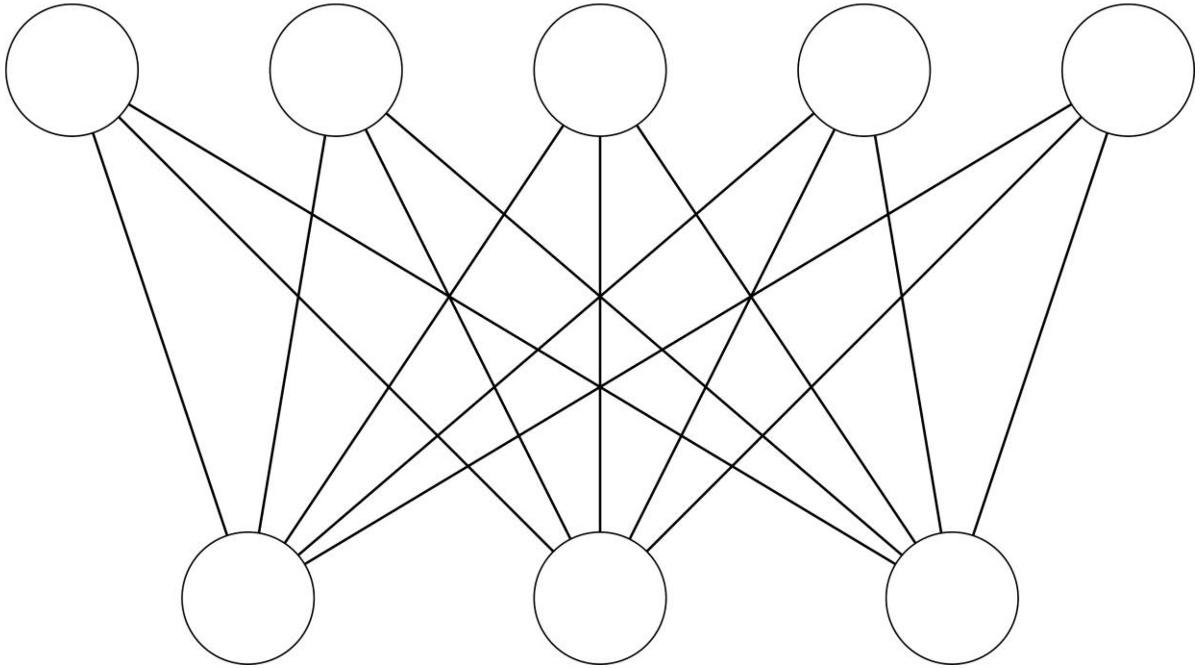


given a graph,
how many successive vertex orderings are there?



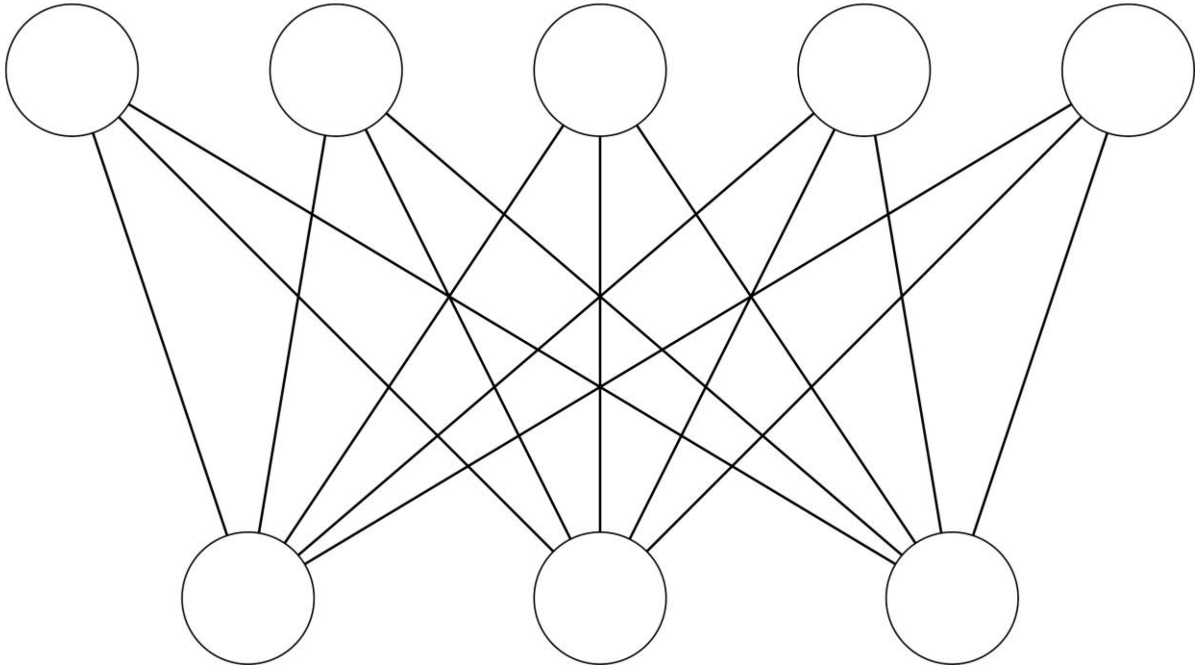
$$\sigma(K_n) = n!$$

given a graph,
how many successive vertex orderings are there?

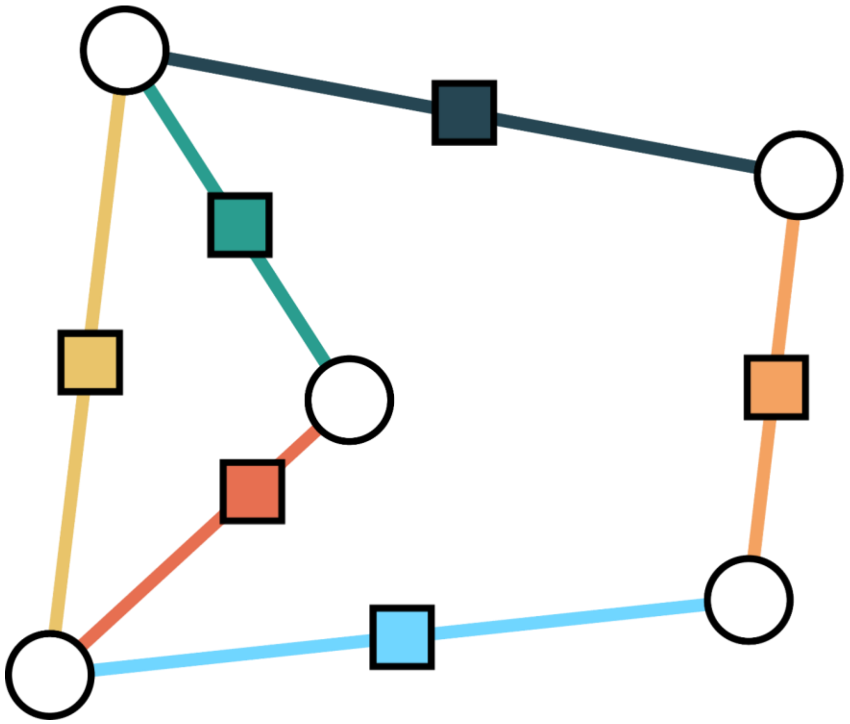


$$\sigma(K_{m,n}) = 2mn(m+n-2)!$$

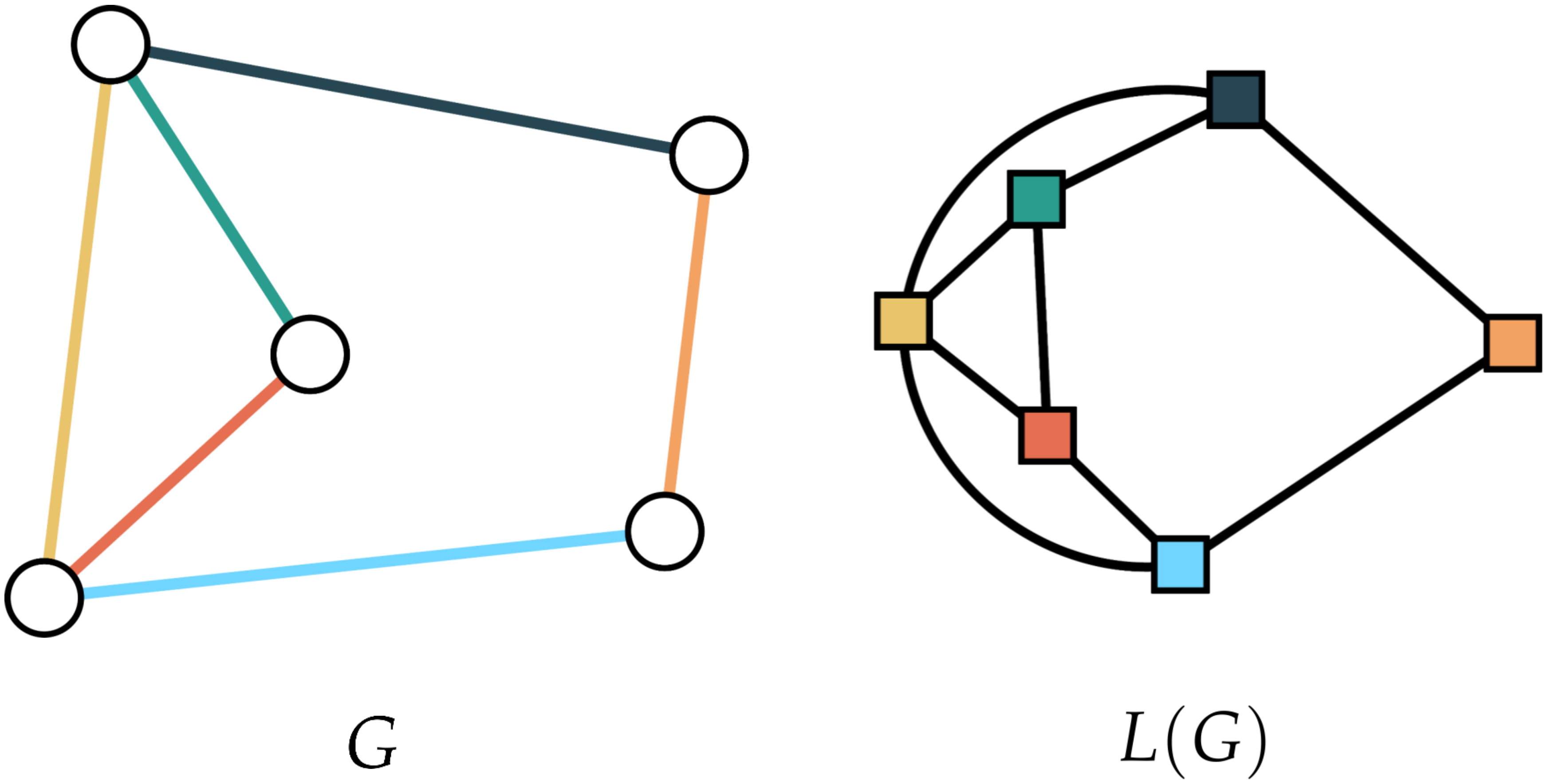
given a graph,
how many successive *edge* orderings are there?



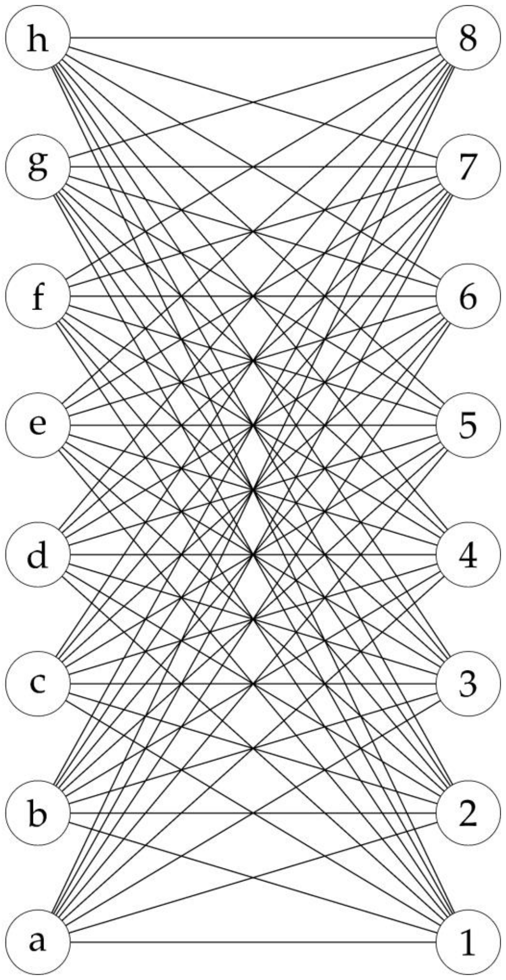
line graphs



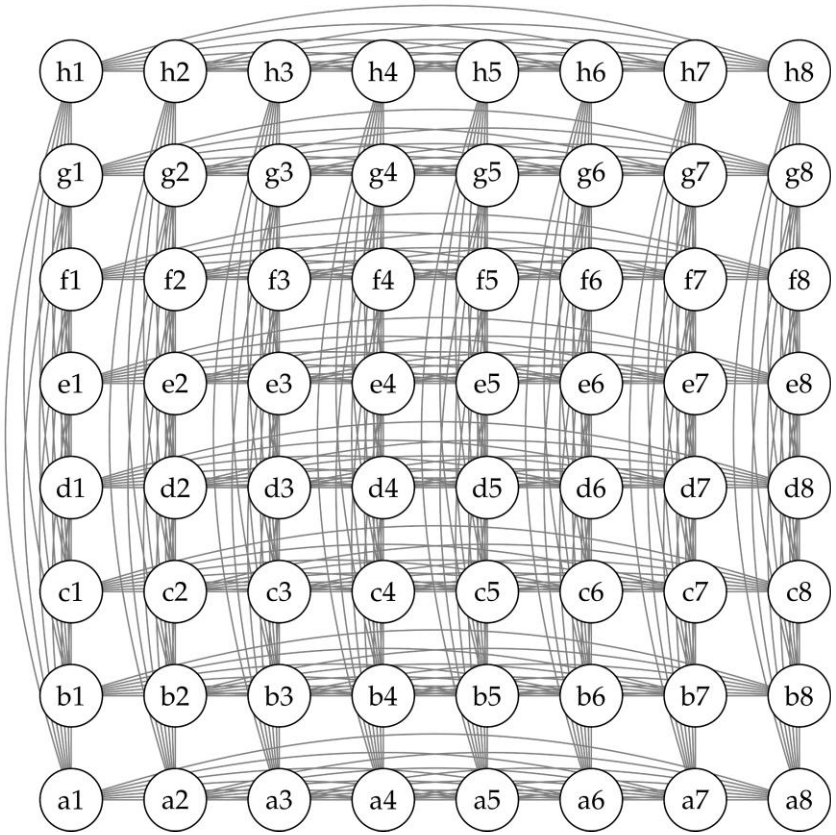
line graphs



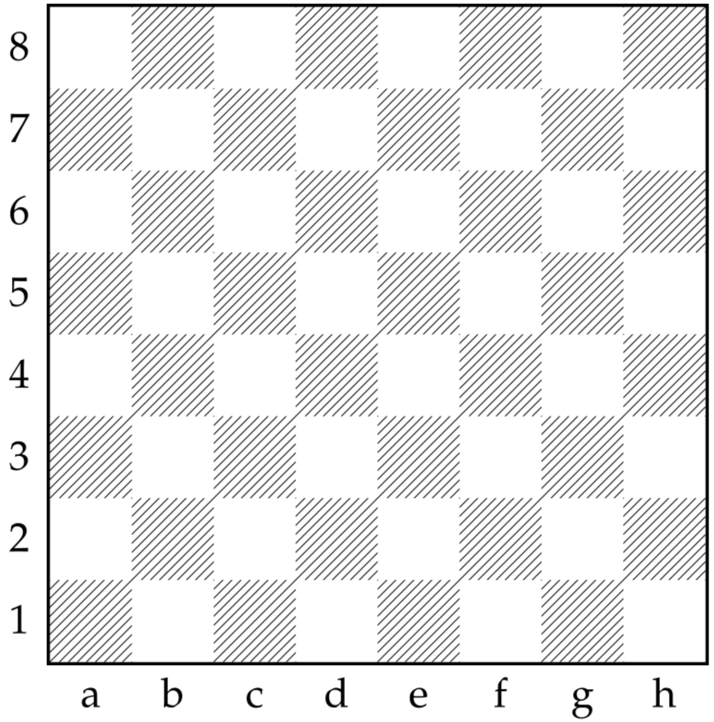
line graphs



$K_{m,n}$



$L(K_{m,n})$

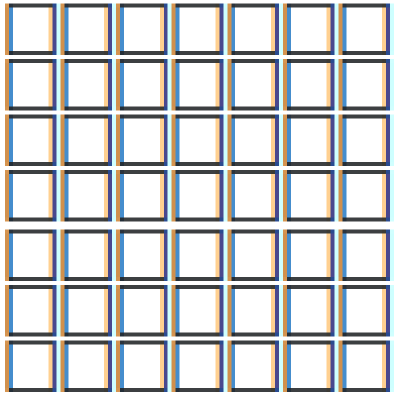


rook graph

successive edge orderings of G
$=$ successive vertex orderings of $L(G)$

Gao–Peng, 2021

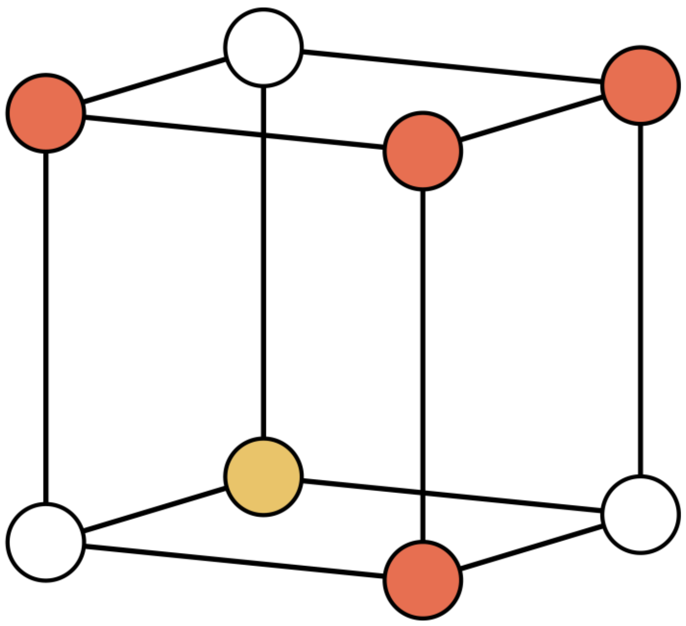
$$\sigma\big(L(K_{m,n})\big) = (mn)! \cdot \frac{m+n}{\binom{m+n}{m}}$$



Stanley, 2018

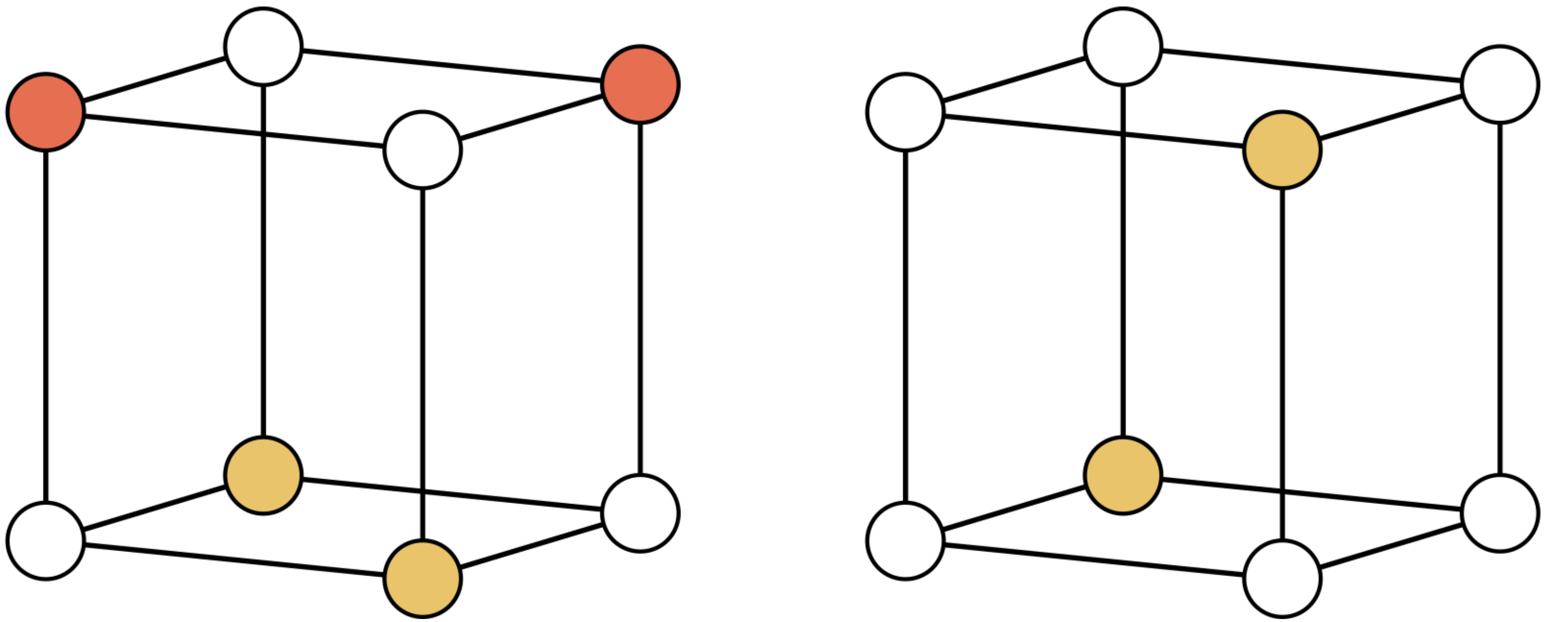
$$\sigma\big(L(K_n)\big) = \frac{2^n n!(n-1)!\binom{n}{2}!}{4(2n-2)!}$$

a graph is *regular* if for every $v_1 \in V$,
the number of vertices in $V \setminus \{v_1\}$ that are not connected to v_1 is a constant a_1



$$a_1 = 4$$

a graph is *regular* if for every $v_1 \in V$,
the number of vertices in $V \setminus \{v_1\}$ that are not connected to v_1 is a constant a_1



a graph is *fully regular* if for every independent set $I = \{v_1, \ldots, v_j\} \subset V$,
the number of vertices in $V \setminus I$ that are not connected to I is a constant a_j

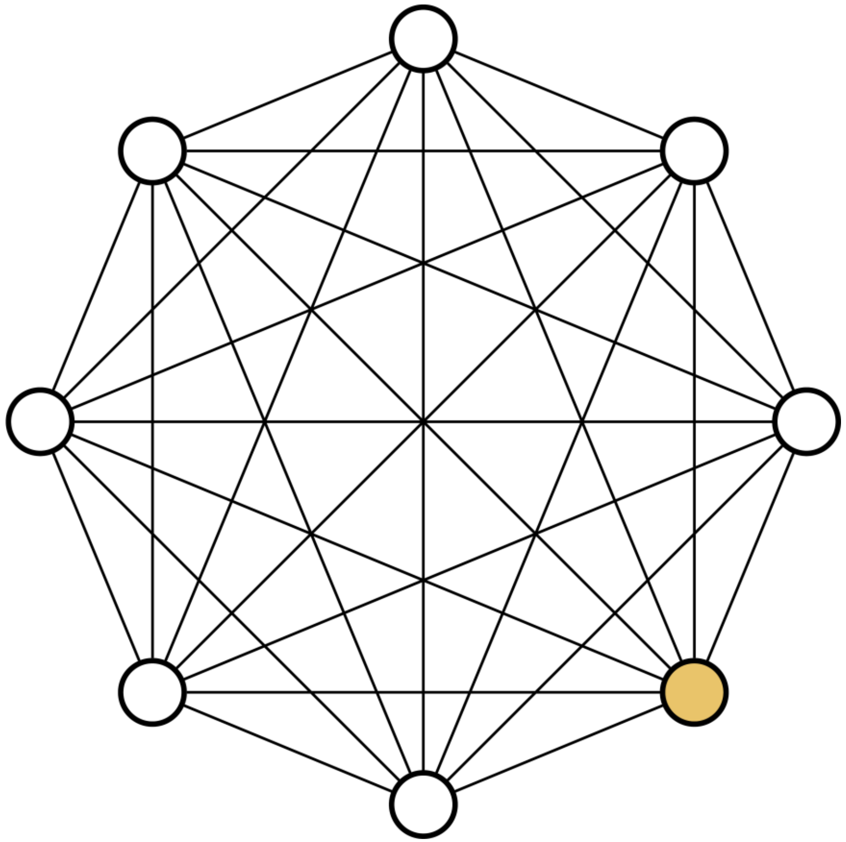
Theorem (Fang–Huang–Pach–Tardos–Zuo, 2023). *The number of successive vertex orderings of a fully regular graph G is given by*

$$\sigma(G) = a_0! \sum_{i=0}^{\alpha} \prod_{j=1}^{i} \frac{-a_j}{a_0 - a_j}$$

Proof idea. Use inclusion-exclusion; the fully regular property makes $\sum_{|I|=m} \Pr(\bigwedge_{v \in I} X_v)$ nice. $\square$

a graph is *fully regular* if for every independent set $I = \{v_1, \ldots, v_j\} \subset V$, the number of vertices in $V \setminus I$ that are not connected to I is a constant a_j

complete graph K_n



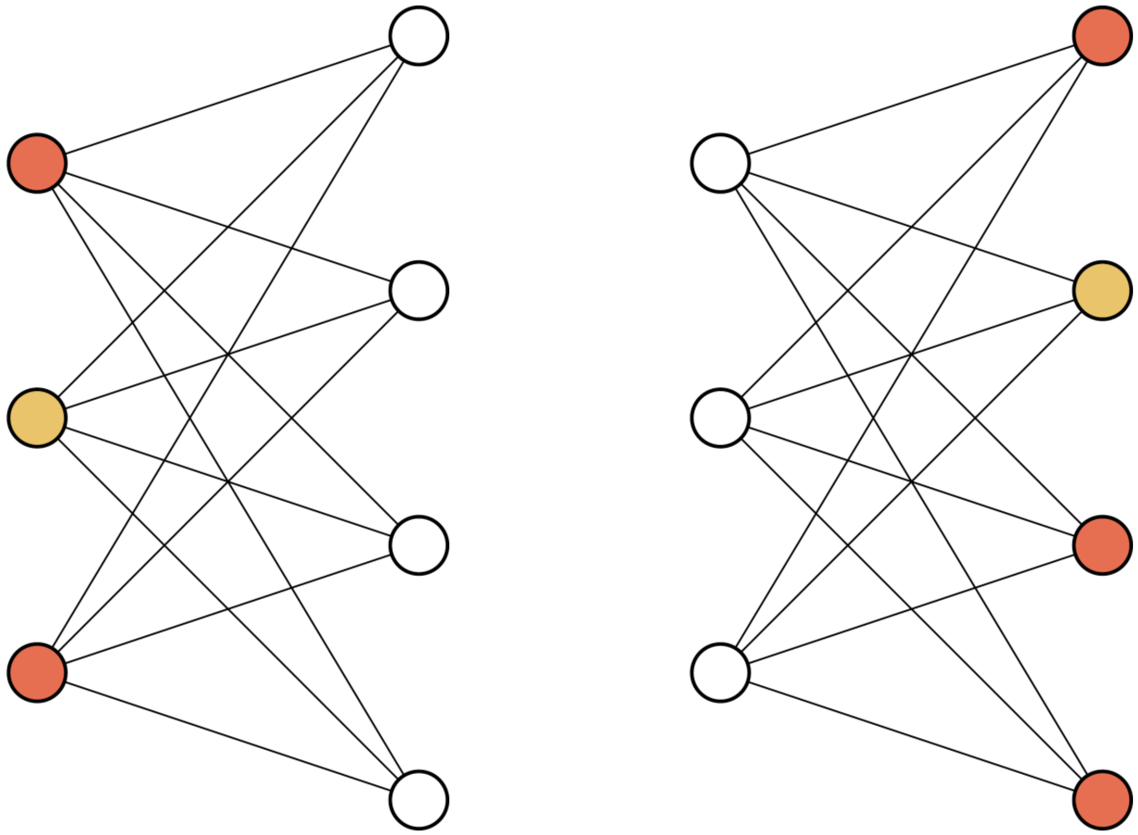
$$a_0 = n$$
$$a_1 = 0$$

$$\sigma(K_n) = n!$$

a graph is *fully regular* if for every independent set $I = \{v_1, \ldots, v_j\} \subset V$, the number of vertices in $V \setminus I$ that are not connected to I is a constant a_j

$$\sigma(G) = a_0! \sum_{i=0}^{\alpha} \prod_{j=1}^{i} \frac{-a_j}{a_0 - a_j}$$

complete bipartite graph $K_{m,n}$



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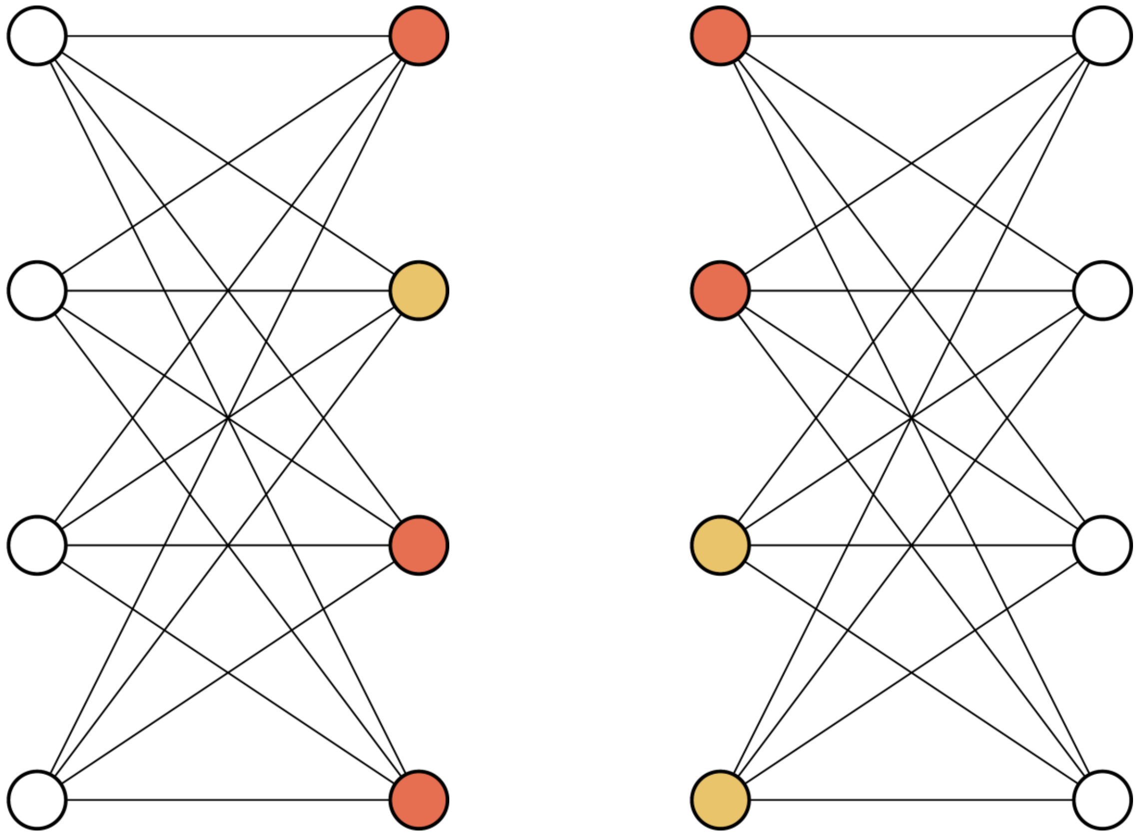
complete bipartite graph $K_{n,n}$



$$a_0 = 2n$$
$$a_1 = n - 1$$
$$\vdots$$
$$a_j = n - j$$

$$\sigma(K_{n,n}) = (2n)! \sum_{i=0}^{n} \prod_{j=1}^{i} \frac{j-n}{j+n}$$

a graph is *fully regular* if for every independent set $I = \{v_1, \ldots, v_j\} \subset V$, the number of vertices in $V \setminus I$ that are not connected to I is a constant a_j

$$\sigma(G) = a_0! \sum_{i=0}^{\alpha} \prod_{j=1}^{i} \frac{-a_j}{a_0 - a_j}$$

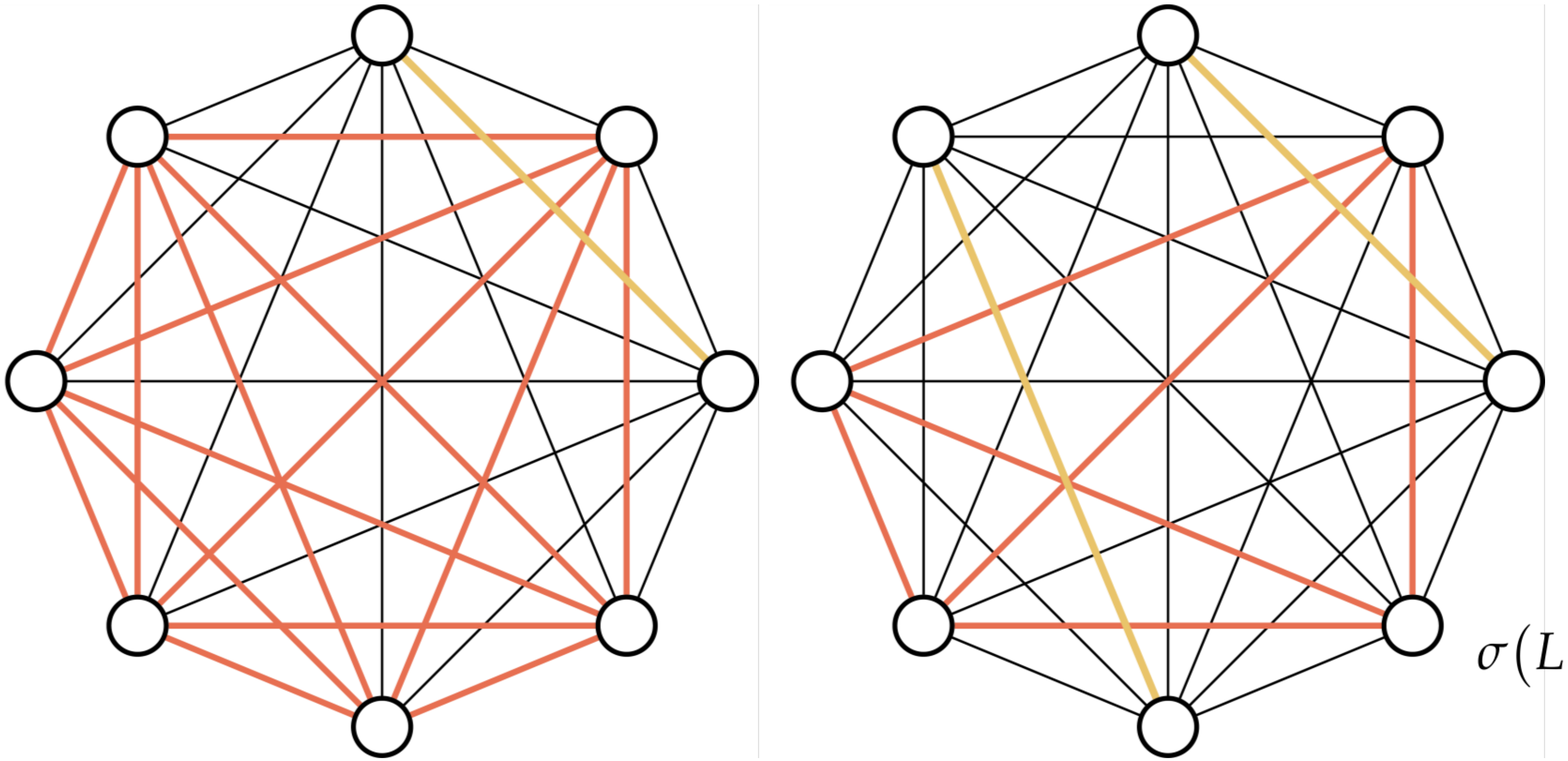
complete bipartite graph $K_{n,n}$

$$\sigma(K_{n,n}) = (2n)! \sum_{i=0}^{n} \prod_{j=1}^{i} \frac{j-n}{j+n}$$
$$= (2n)! \left(1 + \frac{1-n}{1+n} + \frac{1-n}{1+n} \cdot \frac{2-n}{2+n} + \cdots\right)$$
$$= (2n)!\, {}_2F_1\left(\begin{matrix} 1-n,\ 1 \\ 1+n \end{matrix} \,\middle|\, 1\right) = (2n)! \frac{n}{2n-1}$$

a graph is *fully regular* if for every independent set $I = \{v_1, \ldots, v_j\} \subset V$, the number of vertices in $V \setminus I$ that are not connected to I is a constant a_j

$$\sigma(G) = a_0! \sum_{i=0}^{\alpha} \prod_{j=1}^{i} \frac{-a_j}{a_0 - a_j}$$

line graph $L(K_n)$



$$a_0 = \binom{n}{2}$$

$$a_1 = \binom{n-2}{2}$$

$$\vdots$$

$$a_j = \binom{n-2j}{2}$$

$$\sigma(L(K_n)) = \binom{n}{2}! \sum_{i=0}^{\lfloor n/2 \rfloor} \prod_{j=1}^{i} \frac{-\binom{n-2j}{2}}{\binom{n}{2} - \binom{n-2j}{2}}$$

a graph is *fully regular* if for every independent set $I = \{v_1, \ldots, v_j\} \subset V$, the number of vertices in $V \setminus I$ that are not connected to I is a constant a_j

$$\sigma(G) = a_0! \sum_{i=0}^{\alpha} \prod_{j=1}^{i} \frac{-a_j}{a_0 - a_j}$$

line graph $L(K_n)$

$$\sigma(L(K_n)) = \binom{n}{2}! \sum_{i=0}^{\lfloor n/2 \rfloor} \prod_{j=1}^{i} \frac{-\binom{n-2j}{2}}{\binom{n}{2} - \binom{n-2j}{2}}$$

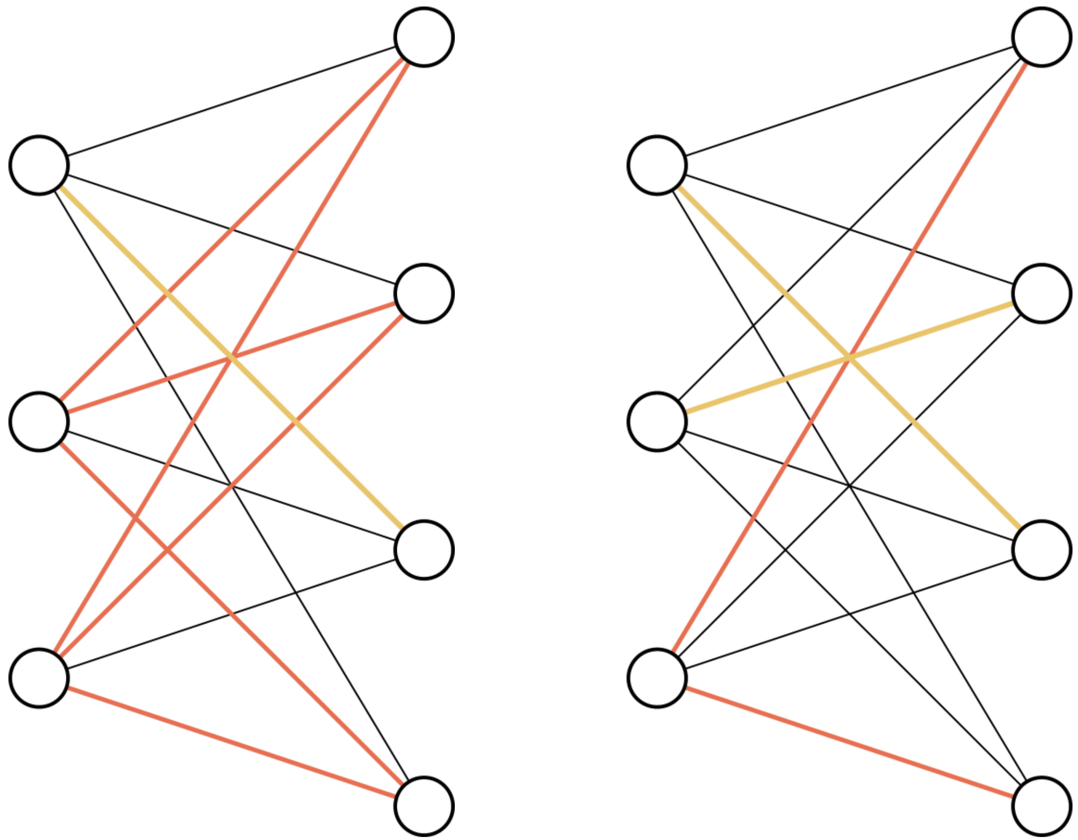
$$= \binom{n}{2}! \left(1 + \frac{\left(1 - \frac{n}{2}\right)\left(\frac{3}{2} - \frac{n}{2}\right)}{1\left(\frac{3}{2} - n\right)} + \frac{\left(1 - \frac{n}{2}\right)\left(\frac{3}{2} - \frac{n}{2}\right)}{1\left(\frac{3}{2} - n\right)} \cdot \frac{\left(2 - \frac{n}{2}\right)\left(\frac{5}{2} - \frac{n}{2}\right)}{2\left(\frac{5}{2} - n\right)} + \cdots \right)$$

$$= \binom{n}{2}!\, {}_2F_1 \left(\left. \begin{matrix} 1 - \frac{n}{2}, \ \frac{3}{2} - \frac{n}{2} \\ \frac{3}{2} - n \end{matrix} \right| 1 \right) = \binom{n}{2}! \frac{2^n n!(n-1)!}{4(2n-2)!} = \binom{n}{2}! \frac{2^{n-2}}{C_{n-1}}$$

a graph is *fully regular* if for every independent set $I = \{v_1, \dots, v_j\} \subset V$, the number of vertices in $V \setminus I$ that are not connected to I is a constant a_j

$$\sigma(G) = a_0! \sum_{i=0}^{\alpha} \prod_{j=1}^{i} \frac{-a_j}{a_0 - a_j}$$

line graph $L(K_{m,n})$



$$a_0 = mn$$

$$a_1 = (m-1)(n-1)$$

$$\vdots$$

$$a_j = (m-j)(n-j)$$

$$\sigma(L(K_{m,n})) = (mn)! \sum_{i=0}^{\min(m,n)} \prod_{j=1}^{i} \frac{-(m-j)(n-j)}{mn-(m-j)(n-j)}$$

a graph is *fully regular* if for every independent set $I = \{v_1, \ldots, v_j\} \subset V$, the number of vertices in $V \setminus I$ that are not connected to I is a constant a_j

$$\sigma(G) = a_0! \sum_{i=0}^{\alpha} \prod_{j=1}^{i} \frac{-a_j}{a_0 - a_j}$$

line graph $L(K_{m,n})$

$$\sigma(L(K_{m,n})) = (mn)! \sum_{i=0}^{\min(m,n)} \prod_{j=1}^{i} \frac{-(m-j)(n-j)}{mn-(m-j)(n-j)}$$

$$= (mn)!\left(1 + \frac{(1-m)(1-n)}{1(1-m-n)} + \frac{(1-m)(1-n)}{1(1-m-n)} \cdot \frac{(2-m)(2-n)}{2(2-m-n)} + \cdots\right)$$

$$= (mn)!\, {}_2F_1\left(\begin{matrix} 1-m,\ 1-n \\ 1-m-n \end{matrix}\,\middle|\, 1\right)$$

$$= (mn)! \frac{m!}{(n+1)\dots(m+n-1)} = (mn)! \frac{m+n}{\binom{m+n}{m}}$$

a graph is *fully regular* if for every independent set $I = \{v_1, \ldots, v_j\} \subset V$, the number of vertices in $V \setminus I$ that are not connected to I is a constant a_j

$$\sigma(G) = a_0! \sum_{i=0}^{\alpha} \prod_{j=1}^{i} \frac{-a_j}{a_0 - a_j}$$

a series $\sum_{k \geq 0} t_k$ is *geometric* if $\frac{t_{k+1}}{t_k} = c$ is a constant function of k

$$t_k = c t_{k-1} = c^2 t_{k-2} = \cdots = c^k t_0$$

$$\sum_{k \geq 0} t_k = t_0 \sum_{k \geq 0} c^k$$

a series $\sum_{k\geq 0} t_k$ is *hypergeometric* if $\frac{t_{k+1}}{t_k} = \frac{P(k)}{Q(k)}$ is a rational function of k

$$\frac{P(k)}{Q(k)} = \frac{(k+a_1)\ldots(k+a_p)}{(k+b_1)\ldots(k+b_q)(k+1)}x$$

$$t_k = \frac{P(k-1)}{Q(k-1)}t_{k-1} = \cdots = \frac{P(k-1)\ldots P(0)}{Q(k-1)\ldots Q(0)}t_0$$

$$\sum_{k\geq 0} t_k = \sum_{k\geq 0} \frac{(a_1)_k\ldots(a_p)_k}{(b_1)_k\ldots(b_q)_k}\frac{x^k}{k!}$$

$$= {}_pF_q\left(\begin{matrix} a_1, & \ldots, & a_p \\ b_1, & \ldots, & b_q \end{matrix} \,\middle|\, x\right)$$

$t_0 = 1$

$(a)_k = a(a+1)\ldots(a+k-1)$

$$e^x = \sum_{k \geq 0} \frac{x^k}{k!} = {}_0F_0\left(\middle| \, x \right)$$

$\sum_{k\geq 0} t_k$ is *hypergeometric* if $\frac{t_{k+1}}{t_k} = \frac{(k+a_1)\ldots(k+a_p)}{(k+b_1)\ldots(k+b_q)(k+1)} x$ is a rational function of k

$$\sum_{k \geq 0} t_k = \sum_{k \geq 0} \frac{(a_1)_k \ldots (a_p)_k}{(b_1)_k \ldots (b_q)_k} \frac{x^k}{k!} = {}_pF_q\left(\begin{matrix} a_1, & \ldots, & a_p \\ b_1, & \ldots, & b_q \end{matrix} \middle| \, x \right)$$

$$(a)_k = a(a+1)\ldots(a+k-1)$$

$$(1-x)^{-a} = \sum_{k\geq 0} \binom{-a}{k} (-x)^k = \sum_{k\geq 0} \frac{(-a)(-a-1)\dots(-a-k+1)}{k!}(-1)^k x^k$$

$$= \sum_{k\geq 0} (a)_k \frac{x^k}{k!} = {}_1F_0\left(\begin{matrix} a \\ \end{matrix} \;\middle|\; x \right)$$

$\sum_{k\geq 0} t_k$ is *hypergeometric* if $\frac{t_{k+1}}{t_k} = \frac{(k+a_1)\dots(k+a_p)}{(k+b_1)\dots(k+b_q)(k+1)} x$ is a rational function of k

$$\sum_{k\geq 0} t_k = \sum_{k\geq 0} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{x^k}{k!} = {}_pF_q\left(\begin{matrix} a_1, & \dots, & a_p \\ b_1, & \dots, & b_q \end{matrix} \;\middle|\; x \right)$$

$$(a)_k = a(a+1)\dots(a+k-1)$$

$$\sin x = \sum_{k\geq 0} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x \sum_{k\geq 0} \frac{(-1)^k x^{2k}}{(2k+1)!} = x\, {}_0F_1\left(\left. {3/2} \,\right|\, -\frac{x^2}{4} \right)$$

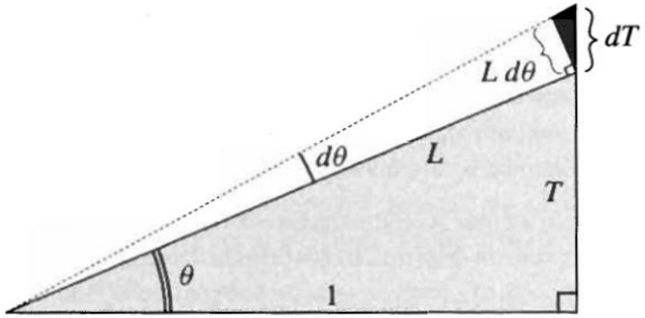
$$\frac{t_{k+1}}{t_k} = \frac{(-1)^{k+1} x^{2k+2}/(2k+3)!}{(-1)^k x^{2k}/(2k+1)!} = \frac{1}{(k+3/2)(k+1)} \left(-\frac{x^2}{4}\right)$$

$\sum_{k\geq 0} t_k$ is *hypergeometric* if $\frac{t_{k+1}}{t_k} = \frac{(k+a_1)\ldots(k+a_p)}{(k+b_1)\ldots(k+b_q)(k+1)} x$ is a rational function of k

$$\sum_{k\geq 0} t_k = \sum_{k\geq 0} \frac{(a_1)_k \ldots (a_p)_k}{(b_1)_k \ldots (b_q)_k} \frac{x^k}{k!} = {}_pF_q\left(\left. \begin{matrix} a_1, & \ldots, & a_p \\ b_1, & \ldots, & b_q \end{matrix} \,\right|\, x \right)$$

$$(a)_k = a(a+1)\ldots(a+k-1)$$

$$\arctan x = \int_0^x \frac{dt}{1+t^2} = \int_0^x \sum_{k\geq 0} (-1)^k t^{2k}\, dt = \sum_{k\geq 0} \frac{(-1)^k x^{2k+1}}{2k+1} = x \sum_{k\geq 0} \frac{(-1)^k x^{2k}}{2k+1}$$



$$= x\, {}_2F_1\left(\begin{matrix} 1/2,\ 1 \\ 3/2 \end{matrix} \,\middle|\, -x^2\right)$$

$$\frac{dT}{L\,d\theta} = \frac{L}{1} \quad \Longrightarrow \quad \frac{dT}{d\theta} = L^2 = 1 + T^2.$$

$$\frac{t_{k+1}}{t_k} = \frac{(-1)^{k+1} x^{2k+2}/(2k+3)}{(-1)^k x^{2k}/(2k+1)} = \frac{k+1/2}{k+3/2}(-x^2) = \frac{(k+1/2)(k+1)}{(k+3/2)(k+1)}(-x^2)$$

$\sum_{k\geq 0} t_k$ is *hypergeometric* if $\frac{t_{k+1}}{t_k} = \frac{(k+a_1)\ldots(k+a_p)}{(k+b_1)\ldots(k+b_q)(k+1)} x$ is a rational function of k

$$\sum_{k\geq 0} t_k = \sum_{k\geq 0} \frac{(a_1)_k \ldots (a_p)_k}{(b_1)_k \ldots (b_q)_k} \frac{x^k}{k!} = {}_pF_q\left(\begin{matrix} a_1, & \ldots, & a_p \\ b_1, & \ldots, & b_q \end{matrix} \,\middle|\, x\right)$$

$$(a)_k = a(a+1)\ldots(a+k-1)$$

Bessel function	$J_\alpha(x) = \sum_{m\geq 0} \frac{(-1)^m}{m!\Gamma(m+\alpha+1)}\left(\frac{x}{2}\right)^{2m+\alpha} = \frac{(x/2)^\alpha}{\Gamma(\alpha+1)}\,{}_0F_1\left(\begin{matrix} \\ \alpha+1\end{matrix}\,\middle|\, -\frac{x^2}{4}\right)$	
incomplete gamma function	$\gamma(s,x) = \int_0^x t^{s-1}e^{-t}\,dt = \frac{x^s}{s}\,{}_1F_1\left(\begin{matrix} s \\ s+1\end{matrix}\,\middle|\, -x\right)$	
Chebyshev polynomial	$T_n(\cos\theta) = \cos n\theta$	$T_n(x) = {}_2F_1\left(\begin{matrix} -n,\ n \\ 1/2\end{matrix}\,\middle|\, \frac{1-x}{2}\right)$
dilogarithm	$\mathrm{Li}_2(x) = \sum_{n>0} \frac{x^n}{n^2} = x\,{}_3F_2\left(\begin{matrix} 1,\ 1,\ 1 \\ 2,\ 2\end{matrix}\,\middle|\, x\right)$	
Bring radical	$x^5 + x + a = 0$	$\mathrm{BR}(t) = -a\,{}_4F_3\left(\begin{matrix} \frac{1}{5},\ \frac{2}{5},\ \frac{3}{5},\ \frac{4}{5} \\ \frac{1}{2},\ \frac{3}{4},\ \frac{5}{4}\end{matrix}\,\middle|\, \frac{5^5}{2^8}t^4\right)$

$\sum_{k\geq 0} t_k$ is *hypergeometric* if $\frac{t_{k+1}}{t_k} = \frac{(k+a_1)\ldots(k+a_p)}{(k+b_1)\ldots(k+b_q)(k+1)}x$ is a rational function of k

$$\sum_{k\geq 0} t_k = \sum_{k\geq 0} \frac{(a_1)_k \ldots (a_p)_k}{(b_1)_k \ldots (b_q)_k}\frac{x^k}{k!} = {}_pF_q\left(\begin{matrix} a_1, & \ldots, & a_p \\ b_1, & \ldots, & b_q\end{matrix}\,\middle|\, x\right)$$

$$(a)_k = a(a+1)\ldots(a+k-1)$$

Theorem (Euler). *If* $\Re(c-b) > 0$ *and* $x \in \mathbf{C} \setminus [1,\infty)$, *then*

$$ {}_2F_1\left(\begin{matrix} a,\ b \\ c \end{matrix} \,\middle|\, x\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}\,dt. $$

Proof. We have

$$ \begin{aligned} &\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}\,dt \\ &\quad = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{n\geq 0} \frac{(a)_n}{n!} x^n \int_0^1 t^{n+b-1}(1-t)^{c-b-1}\,dt \\ &\quad = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{n\geq 0} \frac{(a)_n}{n!} x^n \frac{\Gamma(n+b)\Gamma(c-b)}{\Gamma(n+c)} \\ &\quad = \sum_{n\geq 0} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}. \end{aligned} $$

□

$\sum_{k\geq 0} t_k$ is *hypergeometric* if $\frac{t_{k+1}}{t_k} = \frac{(k+a_1)\dots(k+a_p)}{(k+b_1)\dots(k+b_q)(k+1)} x$ is a rational function of k

$$ \sum_{k\geq 0} t_k = \sum_{k\geq 0} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{x^k}{k!} = {}_pF_q\left(\begin{matrix} a_1, & \dots, & a_p \\ b_1, & \dots, & b_q \end{matrix} \,\middle|\, x\right) $$

$(a)_k = a(a+1)\dots(a+k-1)$

Theorem (Euler). *If $\Re(c-b) > 0$ and $x \in \mathbf{C} \setminus [1,\infty)$, then*

$$ {}_2F_1\left(\begin{matrix} a,\ b \\ c \end{matrix} \,\middle|\, x\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}\, dt. $$

Corollary (Gauss, 1812). *If $\Re(c-a-b) > 0$, we have*

$$ \sum_{n\geq 0} \frac{(a)_n (b)_n}{n!(c)_n} = {}_2F_1\left(\begin{matrix} a,\ b \\ c \end{matrix} \,\middle|\, 1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. $$

Proof. Send $x \to 1^-$ to get

$$ \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-a-b-1}\, dt = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. $$

$\square$

$\sum_{k\geq 0} t_k$ is *hypergeometric* if $\frac{t_{k+1}}{t_k} = \frac{(k+a_1)\ldots(k+a_p)}{(k+b_1)\ldots(k+b_q)(k+1)} x$ is a rational function of k

$$ \sum_{k\geq 0} t_k = \sum_{k\geq 0} \frac{(a_1)_k \ldots (a_p)_k}{(b_1)_k \ldots (b_q)_k} \frac{x^k}{k!} = {}_pF_q\left(\begin{matrix} a_1, & \ldots, & a_p \\ b_1, & \ldots, & b_q \end{matrix} \,\middle|\, x\right) $$

$(a)_k = a(a+1)\ldots(a+k-1)$

$$
\begin{aligned}
{}_2F_1\left(\begin{matrix} a,\ b \\ c \end{matrix}\,\middle|\, x\right) &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 (1-x+xs)^{-a}(1-s)^{b-1}s^{c-b-1}\,ds \\
&= \frac{(1-x)^{-a}\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 \left(1-\frac{xs}{x-1}\right)^{-a} s^{c-b-1}(1-s)^{b-1}\,ds \\
&= (1-x)^{-a}\,{}_2F_1\left(\begin{matrix} a,\ c-b \\ c \end{matrix}\,\middle|\, \frac{x}{x-1}\right) \\
&= (1-x)^{-a}\left(1-\frac{x}{x-1}\right)^{-c+b}{}_2F_1\left(\begin{matrix} c-a,\ c-b \\ c \end{matrix}\,\middle|\, x\right) \\
&= (1-x)^{c-a-b}\,{}_2F_1\left(\begin{matrix} c-a,\ c-b \\ c \end{matrix}\,\middle|\, x\right)
\end{aligned}
$$

$$
{}_2F_1\left(\begin{matrix} a,\ b \\ c \end{matrix}\,\middle|\, x\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}\,dt
$$

$$
\sum_{k\geq 0} t_k = \sum_{k\geq 0}\frac{(a_1)_k\dots(a_p)_k}{(b_1)_k\dots(b_q)_k}\frac{x^k}{k!} = {}_pF_q\left(\begin{matrix} a_1,\ \dots,\ a_p \\ b_1,\ \dots,\ b_q \end{matrix}\,\middle|\, x\right)
$$

$$
(a)_k = a(a+1)\dots(a+k-1)
$$

$$ {}_2F_1\left(\begin{matrix} a,\ b \\ c \end{matrix} \,\middle|\, x\right) = (1-x)^{c-a-b}\, {}_2F_1\left(\begin{matrix} c-a,\ c-b \\ c \end{matrix} \,\middle|\, x\right) $$

$$ (1-x)^{f-d-e}\, {}_2F_1\left(\begin{matrix} f-d,\ f-e \\ f \end{matrix} \,\middle|\, x\right) = {}_2F_1\left(\begin{matrix} d,\ e \\ f \end{matrix} \,\middle|\, x\right) $$

$$ {}_2F_1\left(\begin{matrix} a,\ b \\ c \end{matrix} \,\middle|\, x\right) {}_2F_1\left(\begin{matrix} f-d,\ f-e \\ f \end{matrix} \,\middle|\, x\right) = {}_2F_1\left(\begin{matrix} c-a,\ c-b \\ c \end{matrix} \,\middle|\, x\right) {}_2F_1\left(\begin{matrix} d,\ e \\ f \end{matrix} \,\middle|\, x\right) \qquad c-a-b=f-d-e $$

$$ [x^n] = \sum_{k=0}^{n} \frac{(a)_k (b)_k (f-d)_{n-k} (f-e)_{n-k}}{(c)_k k! (f)_{n-k} (n-k)!} = \frac{(f-d)_n (f-e)_n}{n! (f)_n}\, {}_4F_3\left(\begin{matrix} a,\ b,\ 1-f-n,\ -n \\ c,\ d-f-n+1,\ e-f-n+1 \end{matrix} \,\middle|\, 1\right) $$

$$ {}_4F_3\left(\begin{matrix} -n,\ a,\ b,\ -f-n+1 \\ c,\ d-f-n+1,\ e-f-n+1 \end{matrix} \,\middle|\, 1\right) = \frac{(d)_n (e)_n}{(f-d)_n (f-e)_n}\, {}_4F_3\left(\begin{matrix} -n,\ c-a,\ c-b,\ 1-f-n \\ c,\ 1-d-n,\ 1-e-n \end{matrix} \,\middle|\, 1\right) $$

$$ {}_2F_1\left(\begin{matrix} a,\ b \\ c \end{matrix} \,\middle|\, x\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-xt)^{-a}\, dt $$

$$ \sum_{k\geq 0} t_k = \sum_{k \geq 0} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{x^k}{k!} = {}_pF_q\left(\begin{matrix} a_1,\ \dots,\ a_p \\ b_1,\ \dots,\ b_q \end{matrix} \,\middle|\, x\right) $$

$$ (a)_k = a(a+1)\dots(a+k-1) $$

$$ {}_4F_3\left(\begin{matrix} -n,\ a,\ b,\ -f-n+1 \\ c,\ d-f-n+1,\ e-f-n+1 \end{matrix}\,\middle|\, 1\right) = \frac{(d)_n(e)_n}{(f-d)_n(f-e)_n}\, {}_4F_3\left(\begin{matrix} -n,\ c-a,\ c-b,\ 1-f-n \\ c,\ 1-d-n,\ 1-e-n \end{matrix}\,\middle|\, 1\right) $$

$$ {}_4F_3\left(\begin{matrix} -n,\ a,\ b,\ c \\ d,\ e,\ f \end{matrix}\,\middle|\, 1\right) = \frac{(e-a)_n(f-a)_n}{(e)_n(f)_n}\, {}_4F_3\left(\begin{matrix} -n,\ a,\ d-b,\ d-c \\ d,\ a+1-n-e,\ a+1-n-f \end{matrix}\,\middle|\, 1\right) \qquad \begin{matrix} a+b+c-n+1 \\ =d+e+f \end{matrix} $$

$$ {}_3F_2\left(\begin{matrix} -n,\ a,\ b \\ d,\ e \end{matrix}\,\middle|\, 1\right) = \frac{(e-a)_n}{(e)_n}\, {}_3F_2\left(\begin{matrix} -n,\ a,\ d-b \\ d,\ a+1-n-e \end{matrix}\,\middle|\, 1\right) \qquad \begin{matrix} f, c \to \infty \\ f-c \text{ fixed} \end{matrix} $$

$$ = \frac{(d-a)_n(e-a)_n}{(d)_n(e)_n}\, {}_3F_2\left(\begin{matrix} -n,\ a,\ a+b-n-d-e+1 \\ a-n-d+1,\ a-n-e+1 \end{matrix}\,\middle|\, 1\right) $$

Theorem (Sheppard, 1912). *Given $a, b, d, e \in \mathbf{C}$ and integer $n \geq 0$, we have*

$$ {}_3F_2\left(\begin{matrix} -n,\ a,\ b \\ d,\ e \end{matrix}\,\middle|\, 1\right) = \frac{(d-a)_n(e-a)_n}{(d)_n(e)_n}\, {}_3F_2\left(\begin{matrix} -n,\ a,\ a+b-n-d-e+1 \\ a-n-d+1,\ a-n-e+1 \end{matrix}\,\middle|\, 1\right). $$

$$ {}_2F_1\left(\begin{matrix} a,\ b \\ c \end{matrix}\,\middle|\, x\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}\, dt $$

$$ \sum_{k\geq 0} t_k = \sum_{k\geq 0} \frac{(a_1)_k \ldots (a_p)_k}{(b_1)_k \ldots (b_q)_k} \frac{x^k}{k!} = {}_pF_q\left(\begin{matrix} a_1,\ \ldots,\ a_p \\ b_1,\ \ldots,\ b_q \end{matrix}\,\middle|\, x\right) $$

$$ (a)_k = a(a+1)\ldots(a+k-1) $$

indefinite summation	definite summation
$T_n = \sum_{k=0}^{n} t_k$	$T_n = \sum_{k=-\infty}^{\infty} t_{n,k}$
$\sum_{k=0}^{n} \binom{N}{k}$ no simple form	$\sum_{k=-\infty}^{\infty} \binom{N}{k} = 2^N$
$\int_{-\infty}^{a} e^{-x^2}\, dx$ no simple form	$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$
Gosper's algorithm	Zeilberger's algorithm

Theorem (Sheppard, 1912). *Given $a, b, d, e \in \mathbf{C}$ and integer $n \geq 0$, we have*

$$ {}_3F_2\left(\begin{matrix} -n,\ a,\ b \\ d,\ e \end{matrix} \,\middle|\, 1\right) = \frac{(d-a)_n(e-a)_n}{(d)_n(e)_n}\, {}_3F_2\left(\begin{matrix} -n,\ a,\ a+b-n-d-e+1 \\ a-n-d+1,\ a-n-e+1 \end{matrix} \,\middle|\, 1\right). $$

$$ \sum_{k\geq 0} t_k = \sum_{k\geq 0} \frac{(a_1)_k \ldots (a_p)_k}{(b_1)_k \ldots (b_q)_k} \frac{x^k}{k!} = {}_pF_q\left(\begin{matrix} a_1, & \ldots, & a_p \\ b_1, & \ldots, & b_q \end{matrix} \,\middle|\, x\right) $$

$$ (a)_k = a(a+1)\ldots(a+k-1) $$

Gosper's algorithm

given hypergeometric t_n, does there exist hypergeometric T_n satisfying $T_{n+1} - T_n = t_n$?

(if so, we have $\sum_{k=0}^{n} t_k = T_{n+1} - T_0$)

Problem. *Find a nice form for*

$$\sum_{n=0}^{m} t_n = \sum_{n=0}^{m} \frac{1}{(n+1)4^{2n}} \binom{2n}{n}^2.$$

Solution. By Gosper's algorithm, we have $T_{n+1} - T_n = t_n$, where $T_n = \frac{n}{4^{2n-1}} \binom{2n}{n}^2$. Thus

$$\sum_{n=0}^{m} t_n = T_{m+1} - T_0 = \frac{m+1}{4^{2m+1}} \binom{2m+2}{m+1}^2.$$

$\square$

Theorem (Sheppard, 1912). *Given $a, b, d, e \in \mathbf{C}$ and integer $n \geq 0$, we have*

$${}_3F_2\left(\begin{matrix} -n,\ a,\ b \\ d,\ e \end{matrix} \,\middle|\, 1\right) = \frac{(d-a)_n (e-a)_n}{(d)_n (e)_n} \, {}_3F_2\left(\begin{matrix} -n,\ a,\ a+b-n-d-e+1 \\ a-n-d+1,\ a-n-e+1 \end{matrix} \,\middle|\, 1\right).$$

$$\sum_{k \geq 0} t_k = \sum_{k \geq 0} \frac{(a_1)_k \ldots (a_p)_k}{(b_1)_k \ldots (b_q)_k} \frac{x^k}{k!} = {}_pF_q\left(\begin{matrix} a_1, & \ldots, & a_p \\ b_1, & \ldots, & b_q \end{matrix} \,\middle|\, x\right)$$

$$(a)_k = a(a+1)\ldots(a+k-1)$$

Gosper's algorithm

given hypergeometric t_n, does there exist hypergeometric T_n satisfying $T_{n+1} - T_n = t_n$?

(if so, we have $\sum_{k=0}^{n} t_k = T_{n+1} - T_0$)

Problem. *Find a nice form for*

$$\sum_{n=0}^{m-1} t_n = \sum_{n=0}^{m-1} \frac{2^n}{n+1}.$$

Solution. Gosper's algorithm halts without finding a solution. *This constitutes a rigorous proof that $T_n = \sum_{k=0}^{n-1} t_k$ is not a hypergeometric term.* □

Theorem (Sheppard, 1912). *Given $a, b, d, e \in \mathbf{C}$ and integer $n \geq 0$, we have*

$${}_3F_2\left(\begin{matrix} -n,\ a,\ b \\ d,\ e \end{matrix} \,\middle|\, 1\right) = \frac{(d-a)_n(e-a)_n}{(d)_n(e)_n}\, {}_3F_2\left(\begin{matrix} -n,\ a,\ a+b-n-d-e+1 \\ a-n-d+1,\ a-n-e+1 \end{matrix} \,\middle|\, 1\right).$$

$$\sum_{k\geq 0} t_k = \sum_{k\geq 0} \frac{(a_1)_k \ldots (a_p)_k}{(b_1)_k \ldots (b_q)_k} \frac{x^k}{k!} = {}_pF_q\left(\begin{matrix} a_1, & \ldots, & a_p \\ b_1, & \ldots, & b_q \end{matrix} \,\middle|\, x\right)$$

$$(a)_k = a(a+1)\ldots(a+k-1)$$

Gosper's algorithm

given hypergeometric t_n, does there exist hypergeometric T_n satisfying $T_{n+1} - T_n = t_n$?

(if so, we have $\sum_{k=0}^{n} t_k = T_{n+1} - T_0$)

$$\sum_{k=0}^{m} t_k = \sum_{k=0}^{m} (-1)^k \binom{n}{k} \qquad \text{integer } n > 0$$

1. Write the term ratio as $\frac{t_{k+1}}{t_k} = \frac{p(k+1)}{p(k)} \frac{q(k)}{r(k+1)}$ for polynomials $p(k), q(k), r(k)$ such that if $(k+\alpha) \mid q(k)$ and $(k+\beta) \mid r(k)$, then $\alpha - \beta$ is not a positive integer. (This is always possible and is not hard.)

$$\frac{t_{k+1}}{t_k} = \frac{(-1)^{k+1} n(n-1)\dots(n-k)/(k+1)!}{(-1)^k n(n-1)\dots(n-k+1)/k!} = \frac{k-n}{k+1}$$

$$p(k) = 1, \quad q(k) = k-n, \quad r(k) = k$$

2. Write $T_k = \frac{r(k)s(k)}{p(k)} t_k$. Then the condition $T_{n+1} - T_n = t_n$ implies a recurrence $p(k) = q(k)s(k+1) - r(k)s(k)$ for $s(k)$. (The condition from step 1 and the requirement that T_n is a hypergeometric term force $s(k)$ to be polynomial.) Solve for $s(k)$ to get T_k; if $s(k)$ does not exist, it means that no hypergeometric T_k exists.

$$1 = (k-n)s(k+1) - ks(k)$$

$$s(k) = -1/n$$

$$T_k = \frac{k(-1/n)}{1} (-1)^k \binom{n}{k} = (-1)^{k-1} \binom{n-1}{k-1}$$

Theorem (Sheppard, 1912). *Given $a, b, d, e \in \mathbf{C}$ and integer $n \geq 0$, we have*

$${}_3F_2\left(\begin{matrix} -n,\ a,\ b \\ d,\ e \end{matrix} \,\middle|\, 1\right) = \frac{(d-a)_n (e-a)_n}{(d)_n (e)_n} \, {}_3F_2\left(\begin{matrix} -n,\ a,\ a+b-n-d-e+1 \\ a-n-d+1,\ a-n-e+1 \end{matrix} \,\middle|\, 1\right).$$

$$\sum_{k \geq 0} t_k = \sum_{k \geq 0} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{x^k}{k!} = {}_pF_q\left(\begin{matrix} a_1, & \dots, & a_p \\ b_1, & \dots, & b_q \end{matrix} \,\middle|\, x\right)$$

$$(a)_k = a(a+1)\dots(a+k-1)$$

Gosper's algorithm

given hypergeometric t_n, does there exist hypergeometric T_n satisfying $T_{n+1} - T_n = t_n$?

(if so, we have $\sum_{k=0}^{n} t_k = T_{n+1} - T_0$)

$$\sum_{k=0}^{m} t_k = \sum_{k=0}^{m} \binom{n}{k} \qquad \text{integer } n > 0$$

1. Write the term ratio as $\frac{t_{k+1}}{t_k} = \frac{p(k+1)}{p(k)} \frac{q(k)}{r(k+1)}$ for polynomials $p(k), q(k), r(k)$ such that if $(k+\alpha) \mid q(k)$ and $(k+\beta) \mid r(k)$, then $\alpha - \beta$ is not a positive integer. (This is always possible and is not hard.)

$$\frac{t_{k+1}}{t_k} = \frac{n(n-1)\ldots(n-k)/(k+1)!}{n(n-1)\ldots(n-k+1)/k!} = \frac{n-k}{k+1}$$

$$p(k) = 1, \quad q(k) = n-k, \quad r(k) = k$$

2. Write $T_k = \frac{r(k)s(k)}{p(k)} t_k$. Then the condition $T_{n+1} - T_n = t_n$ implies a recurrence $p(k) = q(k)s(k+1) - r(k)s(k)$ for $s(k)$. (The condition from step 1 and the requirement that T_n is a hypergeometric term force $s(k)$ to be polynomial.) Solve for $s(k)$ to get T_k; if $s(k)$ does not exist, it means that no hypergeometric T_k exists.

$$1 = (n-k)s(k+1) - ks(k)$$

$$2 = (n-2k)\big(s(k+1) + s(k)\big) + n\big(s(k+1) - s(k)\big)$$

$$\deg(s(k)) = d$$

$$d = -1$$

no hypergeometric T_n exists

Theorem (Sheppard, 1912). *Given $a, b, d, e \in \mathbf{C}$ and integer $n \geq 0$, we have*

$$ {}_3F_2\left(\begin{matrix} -n,\ a,\ b \\ d,\ e \end{matrix} \,\middle|\, 1\right) = \frac{(d-a)_n (e-a)_n}{(d)_n (e)_n} \, {}_3F_2\left(\begin{matrix} -n,\ a,\ a+b-n-d-e+1 \\ a-n-d+1,\ a-n-e+1 \end{matrix} \,\middle|\, 1\right).$$

$$\sum_{k \geq 0} t_k = \sum_{k \geq 0} \frac{(a_1)_k \ldots (a_p)_k}{(b_1)_k \ldots (b_q)_k} \frac{x^k}{k!} = {}_pF_q\left(\begin{matrix} a_1,\ \ldots,\ a_p \\ b_1,\ \ldots,\ b_q \end{matrix} \,\middle|\, x\right)$$

$$(a)_k = a(a+1)\ldots(a+k-1)$$

Zeilberger's algorithm

Proposition. *For integer $n \geq 0$, we have*

$$\sum_k \binom{n}{k}^2 = \binom{2n}{n}.$$

Proof. Let $F(n,k) = \binom{n}{k}^2 / \binom{2n}{n}$, and set $G(n,k) := R(n,k)F(n,k)$, where

$$R(n,k) = -\frac{k^2(3n-2k+3)}{2(2n+1)(n-k+1)^2}.$$

Then it is straightforward to verify that

$$F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k);$$

summing both sides over k then gives

$$\sum_k F(n+1,k) = \sum_k F(n,k)$$

for $n \geq 0$. Since $\sum_k F(0,k) = 1$, we are done. $\square$

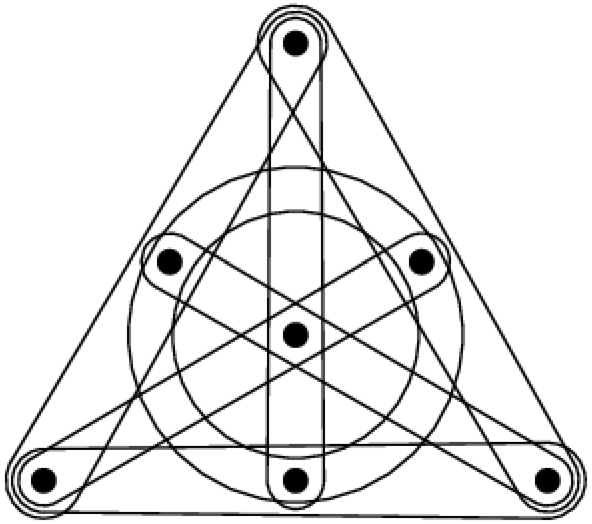
Theorem (Sheppard, 1912). *Given $a, b, d, e \in \mathbf{C}$ and integer $n \geq 0$, we have*

$${}_3F_2\left(\begin{matrix} -n,\ a,\ b \\ d,\ e \end{matrix} \,\middle|\, 1\right) = \frac{(d-a)_n (e-a)_n}{(d)_n (e)_n} \, {}_3F_2\left(\begin{matrix} -n,\ a,\ a+b-n-d-e+1 \\ a-n-d+1,\ a-n-e+1 \end{matrix} \,\middle|\, 1\right).$$

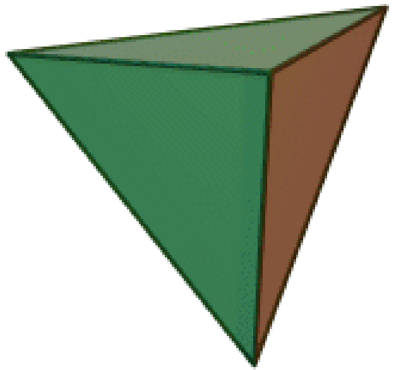
$$\sum_{k \geq 0} t_k = \sum_{k \geq 0} \frac{(a_1)_k \ldots (a_p)_k}{(b_1)_k \ldots (b_q)_k} \frac{x^k}{k!} = {}_pF_q\left(\begin{matrix} a_1, & \ldots, & a_p \\ b_1, & \ldots, & b_q \end{matrix} \,\middle|\, x\right)$$

$$(a)_k = a(a+1)\ldots(a+k-1)$$

An *r-uniform hypergraph* or *r-graph* consists of a finite vertex set V along with an edge set $E \subseteq \binom{V}{r}$. Each edge of the r-graph is an r-element subset of vertices.



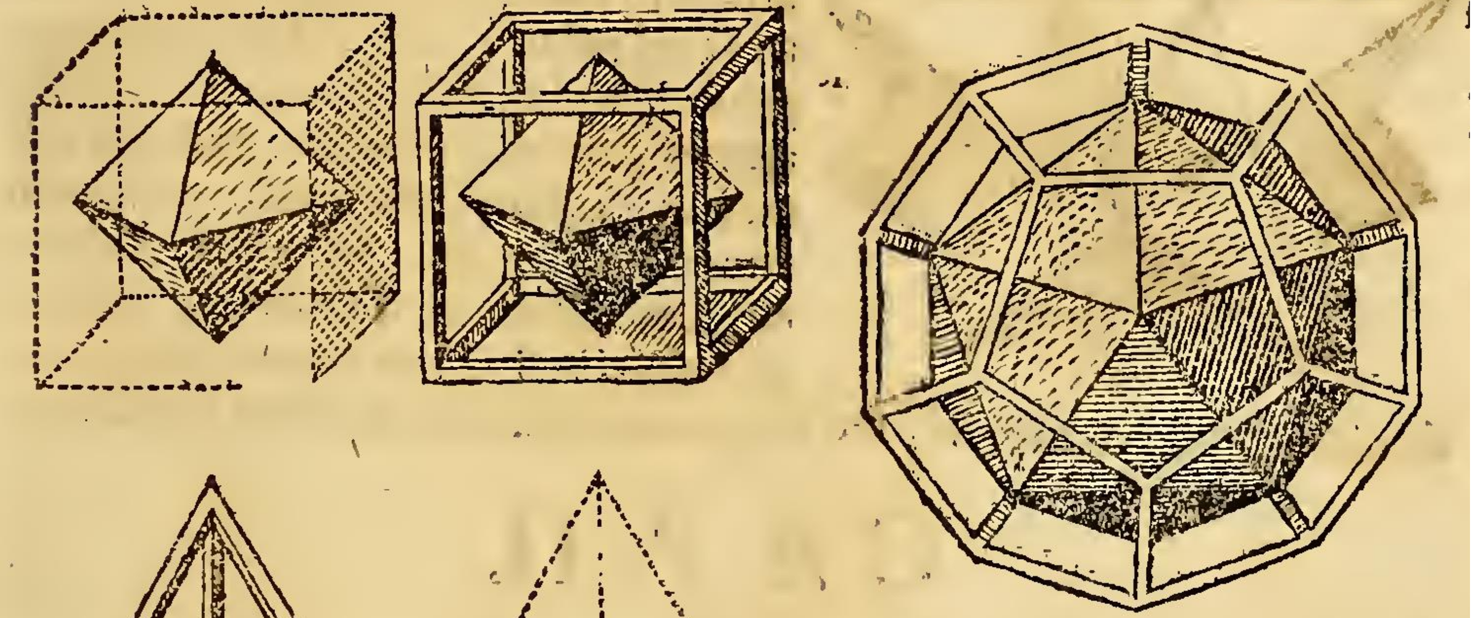
$K_4^{(3)}$ is the complete 3-uniform hypergraph on four vertices

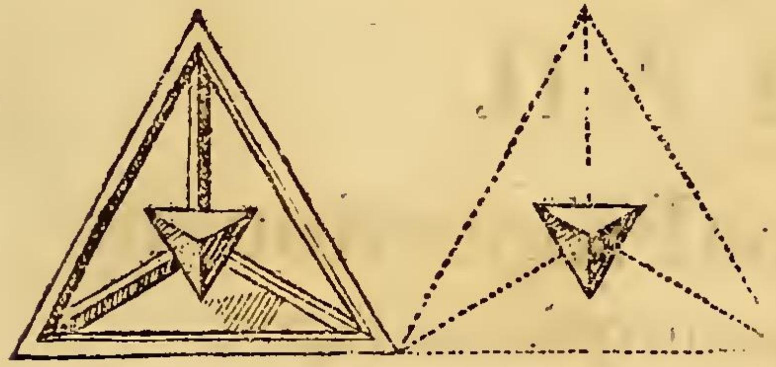


also known as a tetrahedron

the *line graph* $L(H)$ of a hypergraph H has each edge of H as a vertex, with two H-edges adjacent if they intersect

Sunt autem notabilia duo veluti conjugia harum figurarum, ex

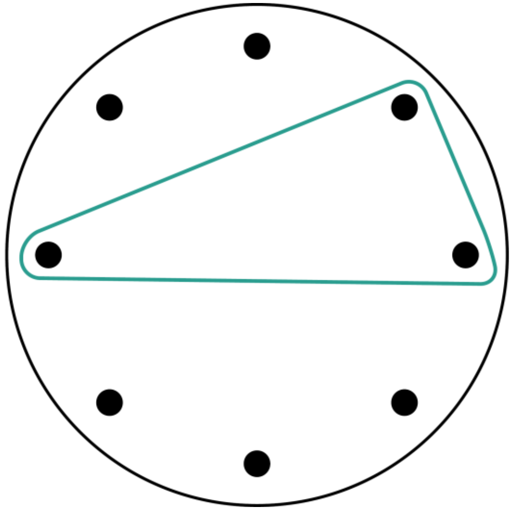




diverſis combinata claſſibus: Mares, Cubus & Dodecaëdron ex primarijs; fœminæ, Octoëdron & Icoſiëdron ex ſecundarijs; quibus accedit una veluti cœlebs aut Androgynos, Tetraëdron; quia ſibi ipſi inſcribitur, ut illæ fœmellæ maribus inſcribuntur & veluti ſubjiciuntur, & ſigna ſexus fœminina maſculinis oppoſita habent, angulos ſcilicet planiciebus.

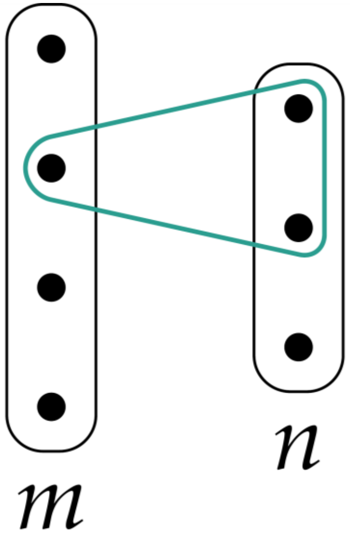
J. Kepler, *Harmonices mundi libri V* (1619), 181

$$K_n^{(3)}$$



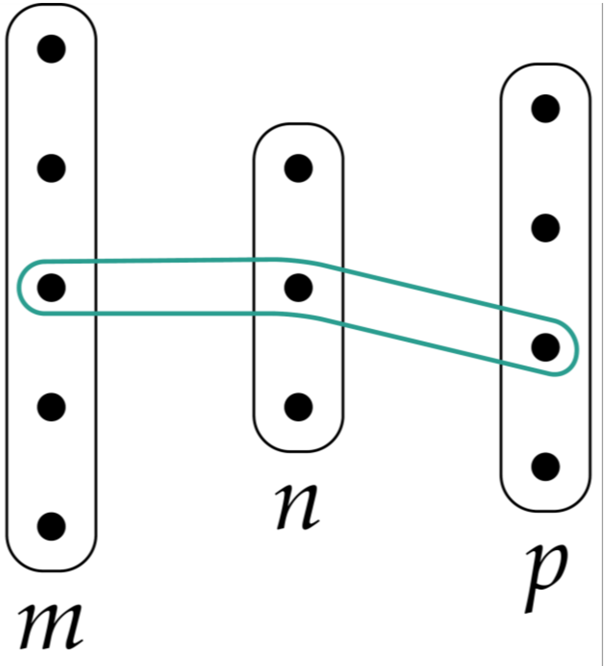
$\binom{n}{3}$ edges

$$K_{m,n}^{(1,2)}$$



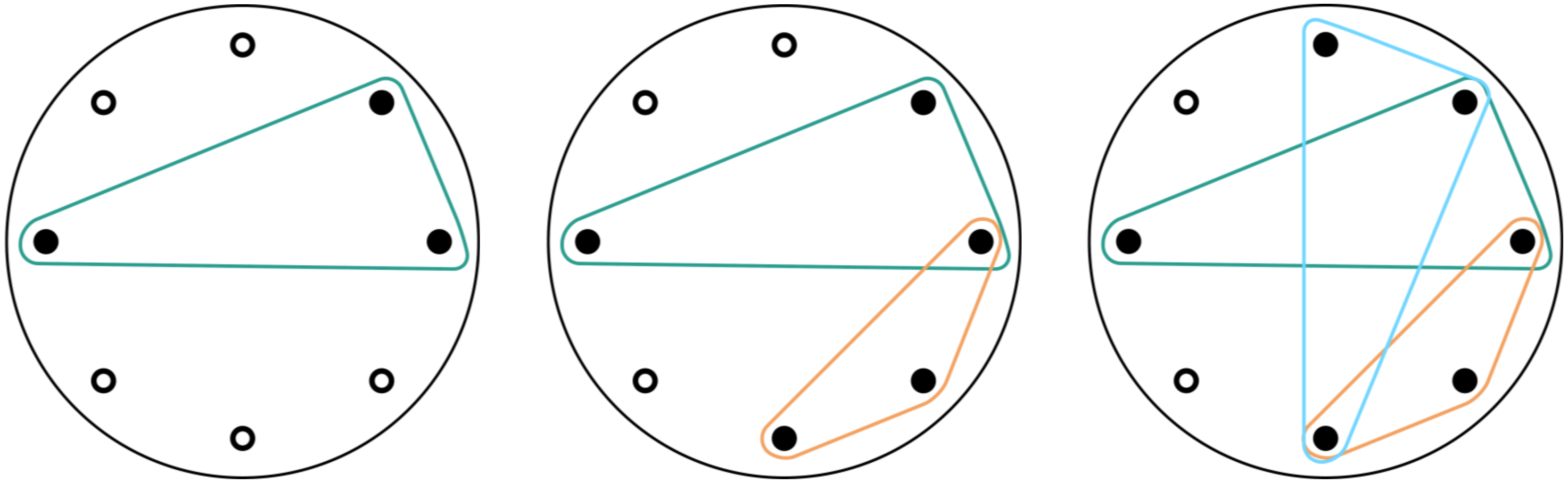
$m\binom{n}{2}$ edges

$$K_{m,n,p}^{(1,1,1)}$$

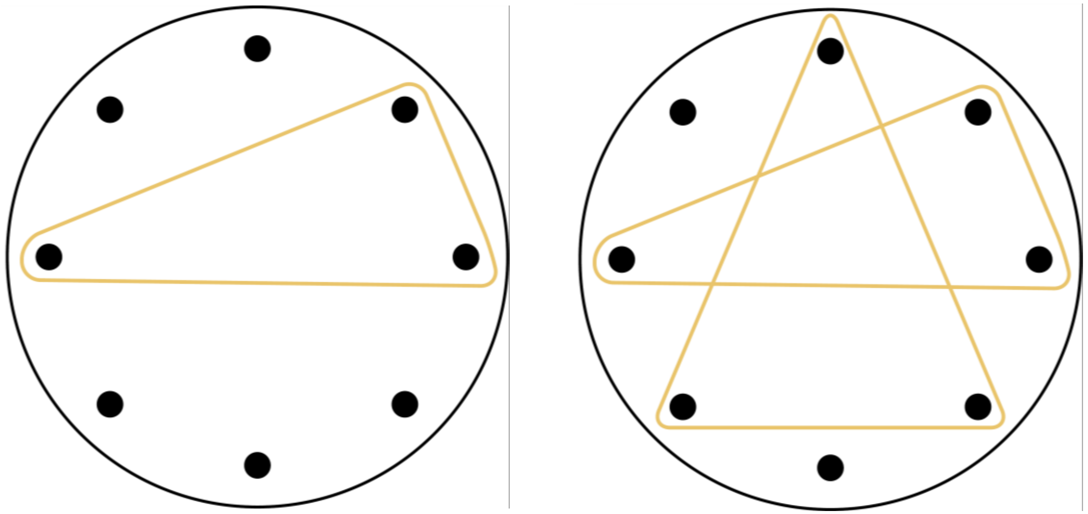


mnp edges

a successive vertex ordering of $L\left(K_n^{(3)}\right)$ is a way to order the 3-element subsets of $\{1, 2, \ldots, n\}$ so that each set after the first one contains some element of a previously chosen set



line graph $L(K_n^{(3)})$



$$a_0 = \binom{n}{3}$$

$$a_1 = \binom{n-3}{3}$$

$$\vdots$$

$$a_j = \binom{n-3j}{3}$$

$$\sigma(L(K_n^{(3)})) = \binom{n}{3}! \sum_{i=0}^{\lfloor n/3 \rfloor} \prod_{j=1}^{i} \frac{-\binom{n-3j}{3}}{\binom{n}{3} - \binom{n-3j}{3}}$$

a graph is *fully regular* if for every independent set $I = \{v_1, \ldots, v_j\} \subset V$, the number of vertices in $V \setminus I$ that are not connected to I is a constant a_j

$$\sigma(G) = a_0! \sum_{i=0}^{\alpha} \prod_{j=1}^{i} \frac{-a_j}{a_0 - a_j}$$

line graph $L(K_n^{(3)})$

$$\sigma(L(K_n^{(3)})) = \binom{n}{3}! \sum_{i=0}^{\lfloor n/3 \rfloor} \prod_{j=1}^{i} \frac{-\binom{n-3j}{3}}{\binom{n}{3} - \binom{n-3j}{3}}$$

```
In[52]:= Table[ Sum[ Product[ -Binomial[n - 3 j, 3] / (Binomial[n, 3] - Binomial[n - 3 j, 3]), {j, 1, i}], {i, 0, Floor[n/3]}], {n, 3, 20}]
```

$$\text{Out[52]=}\ \left\{1, 1, 1, \frac{18}{19}, \frac{27}{31}, \frac{18}{23}, \frac{459}{664}, \frac{297}{493}, \frac{1755}{3379}, \frac{162}{365}, \frac{368631}{979151}, \frac{3726}{11741}, \frac{4363065}{16398253}, \frac{21141}{95215}, \frac{100602}{545179}, \frac{855961182}{5601577967}, \frac{898724539971}{7125324321331}, \frac{308367}{2970431}\right\}$$

```
In[53]:= %52 // FactorInteger

Out[53]= {{{1, 1}}, {{1, 1}}, {{1, 1}}, {{2, 1}, {3, 2}, {19, -1}}, {{3, 3}, {31, -1}}, {{2, 1}, {3, 2}, {23, -1}},
  {{2, -3}, {3, 3}, {17, 1}, {83, -1}}, {{3, 3}, {11, 1}, {17, -1}, {29, -1}}, {{3, 3}, {5, 1}, {13, 1}, {31, -1}, {109, -1}},
  {{2, 1}, {3, 4}, {5, -1}, {73, -1}}, {{3, 5}, {37, 1}, {41, 1}, {47, -1}, {83, -1}, {251, -1}},
  {{2, 1}, {3, 4}, {23, 1}, {59, -1}, {199, -1}}, {{3, 8}, {5, 1}, {7, 1}, {19, 1}, {29, -1}, {47, -1}, {53, -1}, {227, -1}},
  {{3, 6}, {5, -1}, {29, 1}, {137, -1}, {139, -1}}, {{2, 1}, {3, 7}, {23, 1}, {67, -1}, {79, -1}, {103, -1}},
  {{2, 1}, {3, 8}, {19, -1}, {37, 1}, {41, 1}, {43, 1}, {61, -1}, {149, -1}, {163, -1}, {199, -1}},
  {{3, 10}, {41, 1}, {43, 1}, {89, 1}, {97, 1}, {193, -1}, {283, -1}, {409, -1}, {467, -1}, {683, -1}},
  {{3, 8}, {47, 1}, {97, -1}, {113, -1}, {271, -1}}}
```

a graph is *fully regular* if for every independent set $I = \{v_1, \ldots, v_j\} \subset V$, the number of vertices in $V \setminus I$ that are not connected to I is a constant a_j

$$\sigma(G) = a_0! \sum_{i=0}^{\alpha} \prod_{j=1}^{i} \frac{-a_j}{a_0 - a_j}$$

Conjecture (Fang–Huang–Pach–Tardos–Zuo, 2023). *We have, for $n \geq 3$,*

$$\sum_{i=0}^{\lfloor n/3 \rfloor} \prod_{j=1}^{i} \frac{-\binom{n-3j}{3}}{\binom{n}{3} - \binom{n-3j}{3}} = \left\lfloor \frac{n}{3} \right\rfloor \prod_{\substack{j=n+1 \\ 3 \nmid j}}^{\lfloor 3n/2 \rfloor - 2} c_j \Bigg/ \prod_{\substack{j=3 \\ 3 \mid j}}^{n-3} c_j,$$

where $c_j = \frac{6}{j}\left(\binom{n}{3} - \binom{n-j}{3}\right) = j^2 + (3-3n)j + 3n^2 - 6n + 2.$

Theorem (H., 2023). *We have, for $n \geq 3$,*

$$\sum_{i=0}^{\lfloor n/3 \rfloor} \prod_{j=1}^{i} \frac{-\binom{n-3j}{3}}{\binom{n}{3} - \binom{n-3j}{3}} = \left\lfloor \frac{n}{3} \right\rfloor \prod_{\substack{j=n+1 \\ 3 \nmid j}}^{\lfloor 3n/2 \rfloor - 2} c_j \Bigg/ \prod_{\substack{j=3 \\ 3 \mid j}}^{n-3} c_j,$$

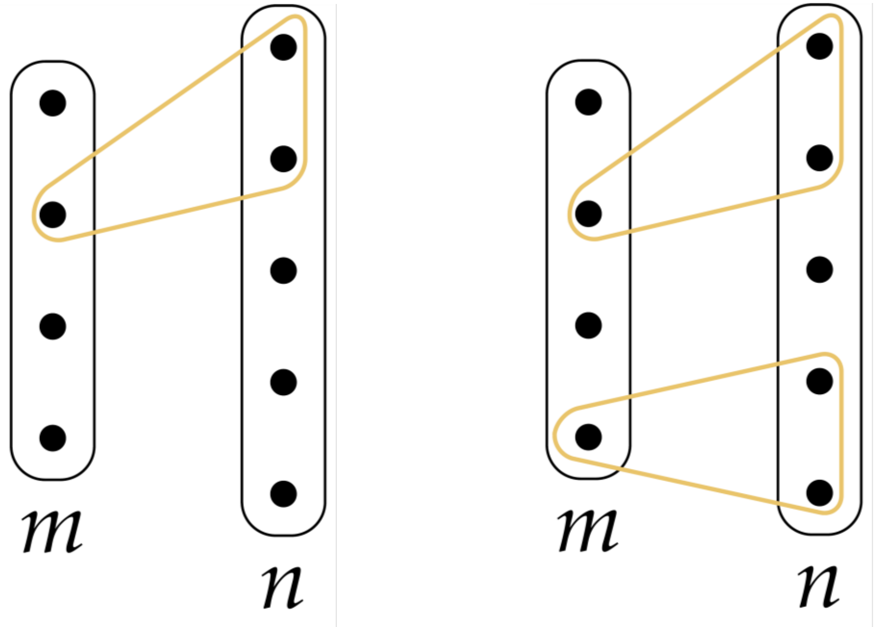
where $c_j = \frac{6}{j}\left(\binom{n}{3} - \binom{n-j}{3}\right) = j^2 + (3-3n)j + 3n^2 - 6n + 2.$

Proof sketch. Let $X_n = \sqrt{1 + 6n - 3n^2}$. Observe that the sum is hypergeometric:

$$\sum_{i=0}^{\lfloor n/3 \rfloor} \prod_{j=1}^{i} \frac{-\binom{n-3j}{3}}{\binom{n}{3} - \binom{n-3j}{3}} = {}_3F_2\left(\begin{matrix} 1 - \frac{n}{3},\ \frac{4}{3} - \frac{n}{3},\ \frac{5}{3} - \frac{n}{3} \\ \frac{1}{6}(9 - 3n + X_n),\ \frac{1}{6}(9 - 3n - X_n) \end{matrix} \,\middle|\, 1 \right).$$

Apply Sheppard's theorem and Gosper's algorithm, then simplify. $\square$

line graph $L(K_{m,n}^{(1,2)})$



$$a_0 = m\binom{n}{2}$$

$$a_1 = (m-1)\binom{n-2}{2}$$

$$\vdots$$

$$a_j = (m-j)\binom{n-2j}{2}$$

$$\sigma(L(K_{m,n}^{(1,2)})) = \left(m\binom{n}{2}\right)! \sum_{i=0}^{\min(m,\lfloor n/2\rfloor)} \prod_{j=1}^{i} \frac{-(m-j)\binom{n-2j}{2}}{m\binom{n}{2} - (m-j)\binom{n-2j}{2}}$$

a graph is *fully regular* if for every independent set $I = \{v_1, \ldots, v_j\} \subset V$, the number of vertices in $V \setminus I$ that are not connected to I is a constant a_j

$$\sigma(G) = a_0! \sum_{i=0}^{\alpha} \prod_{j=1}^{i} \frac{-a_j}{a_0 - a_j}$$

Conjecture (Fang–Huang–Pach–Tardos–Zuo, 2023). *We have, for $m \geq 1$ and $n \geq 2$,*

$$\sum_{i=0}^{\min\{m,\lfloor n/2\rfloor\}} \prod_{j=1}^{i} \frac{-(m-j)\binom{n-2j}{2}}{m\binom{n}{2}-(m-j)\binom{n-2j}{2}} = m \prod_{j=1}^{m-1} \frac{mn-\binom{m+1}{2}+\binom{j}{2}}{d_j},$$

where $d_j = \frac{1}{j}\left(m\binom{n}{2}-(m-j)\binom{n-2j}{2}\right)$, and where any numerators or denominators on the right that are equal to zero are ignored.

Theorem (H., 2023). *We have, for $m \geq 1$ and $n \geq 2$,*

$$\sum_{i=0}^{\min\{m,\lfloor n/2\rfloor\}} \prod_{j=1}^{i} \frac{-(m-j)\binom{n-2j}{2}}{m\binom{n}{2}-(m-j)\binom{n-2j}{2}} = m \prod_{j=1}^{m-1} \frac{mn-\binom{m+1}{2}+\binom{j}{2}}{d_j},$$

where $d_j = \frac{1}{j}\left(m\binom{n}{2}-(m-j)\binom{n-2j}{2}\right)$, and where any numerators or denominators on the right that are equal to zero are ignored.

Proof sketch. Let $Y_{m,n} = \sqrt{(2m+1)^2-8mn}$. Observe that the sum is hypergeometric:

$$\sum_{i=0}^{\min\{m,\lfloor n/2\rfloor\}} \prod_{j=1}^{i} \frac{-(m-j)\binom{n-2j}{2}}{m\binom{n}{2}-(m-j)\binom{n-2j}{2}} = {}_3F_2\left(\begin{matrix} 1-\frac{n}{2},\ \frac{3}{2}-\frac{n}{2},\ 1-m \\ \frac{1}{4}(5-2m-2n+Y_{m,n}),\ \frac{1}{4}(5-2m-2n-Y_{m,n}) \end{matrix} \middle|\, 1\right)$$

Apply Sheppard's theorem and Gosper's algorithm, then simplify carefully. $\square$

```
In[64]:= Table[Sum[Product[-Binomial[n - 4 j, 4]/(Binomial[n, 4] - Binomial[n - 4 j, 4]), {j, 1, i}], {i, 0, Floor[n/4]}], {n, 40, 44}] // FactorInteger

Out[64]= {{{2, 21}, {13, -1}, {23, -1}, {41, -1}, {53, -2}, {61, -1}, {73, -1}, {89, -1},
   {241, -1}, {331, -1}, {743, -1}, {761, -1}, {911, -1}, {1091, -1}, {19154778839631970056937, 1}},
  {{2, 18}, {5, -2}, {13, -1}, {17, -1}, {43, -1}, {47, -1}, {59, -1}, {67, -1}, {71, -1},
   {89, -1}, {157, -1}, {211, -1}, {269, -1}, {557, 1}, {1409, -1}, {1415797460315131, 1}},
  {{2, 18}, {3, -1}, {5, -1}, {7, -4}, {11, -2}, {17, 1}, {23, -1}, {53, -1}, {61, -1}, {73, -1},
   {191, -1}, {331, 1}, {373, -1}, {449, -1}, {829, -1}, {108971, 1}, {207004691, 1}},
  {{2, 20}, {5, -2}, {7, -1}, {29, -1}, {47, -1}, {59, -1}, {67, -1}, {71, -1}, {79, -2}, {131, 1},
   {181, -1}, {311, 1}, {317, -1}, {521, -1}, {1013, -1}, {1091, -1}, {15393787603028743, 1}},
  {{2, 22}, {3, -3}, {7, -1}, {23, -1}, {29, -1}, {31, -1}, {53, -3}, {61, -1}, {71, -1}, {73, -1},
   {179, -1}, {181, -1}, {449, 1}, {499, -1}, {1367, -1}, {265423, 1}, {347851378367, 1}}}
```

```
In[65]:= Sum[Product[-Binomial[n - 4 j, 4]/(Binomial[n, 4] - Binomial[n - 4 j, 4]), {j, 1, i}], {i, 0, Floor[n/4]}, Method -> HypergeometricTermPFQ]
```

$$\text{Out[65]= Sum}\left[\frac{\left(1-\frac{n}{4}\right)_i\left(\frac{5}{4}-\frac{n}{4}\right)_i\left(\frac{3}{2}-\frac{n}{4}\right)_i\left(\frac{7}{4}-\frac{n}{4}\right)_i}{\Gamma(i+1)\left(\frac{7}{4}-\frac{n}{2}\right)_i\left(\frac{1}{8}\left(-2n-\sqrt{-4n^2+12n+1}+11\right)\right)_i\left(\frac{1}{8}\left(-2n+\sqrt{-4n^2+12n+1}+11\right)\right)_i},\ \left\{i,0,\left\lfloor\frac{n}{4}\right\rfloor\right\},\ \text{Method}\to\text{HypergeometricTermPFQ}\right]$$

terminating 0-balanced ${}_4F_3(1)$

$$\text{In[68]:= FullSimplify}\left[\text{Sum}\left[\prod_{j=1}^{i} \frac{-\text{Binomial}[n-5j,5]}{\text{Binomial}[n,5]-\text{Binomial}[n-5j,5]}, \{i, 0, \text{Floor}\left[\frac{n}{5}\right]\}, \text{Method} \to \text{HypergeometricTermPFQ}\right]\right]$$

$$\text{Out[68]= Sum}\Bigg[\left(\text{Pochhammer}\left[1-\frac{n}{5},i\right]\text{Pochhammer}\left[\frac{6}{5}-\frac{n}{5},i\right]\text{Pochhammer}\left[\frac{7}{5}-\frac{n}{5},i\right]\text{Pochhammer}\left[\frac{8}{5}-\frac{n}{5},i\right]\text{Pochhammer}\left[\frac{9}{5}-\frac{n}{5},i\right]\right)\Big/$$

$$\Bigg(\text{Gamma}[1+i]\,\text{Pochhammer}\Bigg[\frac{1}{60}\Bigg(90-15n+\sqrt{3}\sqrt{\left(75(-2+n)^2-40(7-8n+2n^2)+\frac{4(13-100n+185n^2-80n^3+10n^4)}{\left(35-840n+810n^2+500n^3-\frac{1125n^4}{2}+150n^5-\frac{25n^6}{2}+\frac{3}{2}\sqrt{3}\sqrt{-144-1200n+41260n^2+171520n^3-1146240n^4+2320800n^5-2491800n^6+1630800n^7-683925n^8+185000n^9-31250n^{10}+3000n^{11}-125n^{12}}\right)^{1/3}}+2\times2^{2/3}\left(70-1680n+1620n^2+1000n^3-1125n^4+300n^5-25n^6+3\sqrt{3}\sqrt{-144-1200n+41260n^2+171520n^3-1146240n^4+2320800n^5-2491800n^6+1630800n^7-683925n^8+185000n^9-31250n^{10}+3000n^{11}-125n^{12}}\right)^{1/3}\right)}-$$

$$\sqrt{6}\sqrt{\Bigg(75(-2+n)^2-40(7-8n+2n^2)-\frac{2(13-100n+185n^2-80n^3+10n^4)}{\left(35-840n+810n^2+500n^3-\frac{1125n^4}{2}+150n^5-\frac{25n^6}{2}+\frac{3}{2}\sqrt{3}\sqrt{-144-1200n+41260n^2+171520n^3-1146240n^4+2320800n^5-2491800n^6+1630800n^7-683925n^8+185000n^9-31250n^{10}+3000n^{11}-125n^{12}}\right)^{1/3}}-2^{2/3}\left(70-1680n+1620n^2+1000n^3-1125n^4+300n^5-25n^6+3\sqrt{3}\sqrt{-144-1200n+41260n^2+171520n^3-1146240n^4+2320800n^5-2491800n^6+1630800n^7-683925n^8+185000n^9-31250n^{10}+3000n^{11}-125n^{12}}\right)^{1/3}-}$$

$$\left(15\sqrt{3}\,n(8-6n+n^2)\right)\Big/\left(\sqrt{\left(75(-2+n)^2-40(7-8n+2n^2)+\frac{4(13-100n+185n^2-80n^3+10n^4)}{\left(35-840n+810n^2+500n^3-\frac{1125n^4}{2}+150n^5-\frac{25n^6}{2}+\frac{3}{2}\sqrt{3}\sqrt{-144-1200n+41260n^2+171520n^3-1146240n^4+2320800n^5-2491800n^6+1630800n^7-683925n^8+185000n^9-31250n^{10}+3000n^{11}-125n^{12}}\right)^{1/3}}+2\times2^{2/3}\left(70-1680n+1620n^2+1000n^3-1125n^4+300n^5-25n^6+3\sqrt{3}\sqrt{-144-1200n+41260n^2+171520n^3-1146240n^4+2320800n^5-2491800n^6+1630800n^7-683925n^8+185000n^9-31250n^{10}+3000n^{11}-125n^{12}}\right)^{1/3}\right)}\right)\Bigg)\Bigg),i\Bigg]$$

$$\text{Pochhammer}\Bigg[\frac{1}{60}\Bigg(90-15n+\sqrt{3}\sqrt{\left(75(-2+n)^2-40(7-8n+2n^2)+\frac{4(13-100n+185n^2-80n^3+10n^4)}{\left(35-840n+810n^2+500n^3-\frac{1125n^4}{2}+150n^5-\frac{25n^6}{2}+\frac{3}{2}\sqrt{3}\sqrt{-144-1200n+41260n^2+171520n^3-1146240n^4+2320800n^5-2491800n^6+1630800n^7-683925n^8+185000n^9-31250n^{10}+3000n^{11}-125n^{12}}\right)^{1/3}}+2\times2^{2/3}\left(70-1680n+1620n^2+1000n^3-1125n^4+300n^5-25n^6+3\sqrt{3}\sqrt{-144-1200n+41260n^2+171520n^3-1146240n^4+2320800n^5-2491800n^6+1630800n^7-683925n^8+185000n^9-31250n^{10}+3000n^{11}-125n^{12}}\right)^{1/3}\right)}+$$

$$\sqrt{6}\sqrt{\Bigg(75(-2+n)^2-40(7-8n+2n^2)-\frac{2(13-100n+185n^2-80n^3+10n^4)}{\left(35-840n+810n^2+500n^3-\frac{1125n^4}{2}+150n^5-\frac{25n^6}{2}+\frac{3}{2}\sqrt{3}\sqrt{-144-1200n+41260n^2+171520n^3-1146240n^4+2320800n^5-2491800n^6+1630800n^7-683925n^8+185000n^9-31250n^{10}+3000n^{11}-125n^{12}}\right)^{1/3}}-2^{2/3}\left(70-1680n+1620n^2+1000n^3-1125n^4+300n^5-25n^6+3\sqrt{3}\sqrt{-144-1200n+41260n^2+171520n^3-1146240n^4+2320800n^5-2491800n^6+1630800n^7-683925n^8+185000n^9-31250n^{10}+3000n^{11}-125n^{12}}\right)^{1/3}-}$$

$$\left(15\sqrt{3}\,n(8-6n+n^2)\right)\Big/\left(\sqrt{\left(75(-2+n)^2-40(7-8n+2n^2)+\frac{4(13-100n+185n^2-80n^3+10n^4)}{\left(35-840n+810n^2+500n^3-\frac{1125n^4}{2}+150n^5-\frac{25n^6}{2}+\frac{3}{2}\sqrt{3}\sqrt{-144-1200n+41260n^2+171520n^3-1146240n^4+2320800n^5-2491800n^6+1630800n^7-683925n^8+185000n^9-31250n^{10}+3000n^{11}-125n^{12}}\right)^{1/3}}+2\times2^{2/3}\left(70-1680n+1620n^2+1000n^3-1125n^4+300n^5-25n^6+3\sqrt{3}\sqrt{-144-1200n+41260n^2+171520n^3-1146240n^4+2320800n^5-2491800n^6+1630800n^7-683925n^8+185000n^9-31250n^{10}+3000n^{11}-125n^{12}}\right)^{1/3}\right)}\right)\Bigg)\Bigg),i\Bigg]$$

$$\text{Pochhammer}\Bigg[\frac{1}{60}\Bigg(90-15n-\sqrt{3}\sqrt{\left(75(-2+n)^2-40(7-8n+2n^2)+\frac{4(13-100n+185n^2-80n^3+10n^4)}{\left(35-840n+810n^2+500n^3-\frac{1125n^4}{2}+150n^5-\frac{25n^6}{2}+\frac{3}{2}\sqrt{3}\sqrt{-144-1200n+41260n^2+171520n^3-1146240n^4+2320800n^5-2491800n^6+1630800n^7-683925n^8+185000n^9-31250n^{10}+3000n^{11}-125n^{12}}\right)^{1/3}}+2\times2^{2/3}\left(70-1680n+1620n^2+1000n^3-1125n^4+300n^5-25n^6+3\sqrt{3}\sqrt{-144-1200n+41260n^2+171520n^3-1146240n^4+2320800n^5-2491800n^6+1630800n^7-683925n^8+185000n^9-31250n^{10}+3000n^{11}-125n^{12}}\right)^{1/3}\right)}-$$

$$\sqrt{6}\sqrt{\Bigg(75(-2+n)^2-40(7-8n+2n^2)-\frac{2(13-100n+185n^2-80n^3+10n^4)}{\left(35-840n+810n^2+500n^3-\frac{1125n^4}{2}+150n^5-\frac{25n^6}{2}+\frac{3}{2}\sqrt{3}\sqrt{-144-1200n+41260n^2+171520n^3-1146240n^4+2320800n^5-2491800n^6+1630800n^7-683925n^8+185000n^9-31250n^{10}+3000n^{11}-125n^{12}}\right)^{1/3}}-2^{2/3}\left(70-1680n+1620n^2+1000n^3-1125n^4+300n^5-25n^6+3\sqrt{3}\sqrt{-144-1200n+41260n^2+171520n^3-1146240n^4+2320800n^5-2491800n^6+1630800n^7-683925n^8+185000n^9-31250n^{10}+3000n^{11}-125n^{12}}\right)^{1/3}+}$$

$$\left(15\sqrt{3}\,n(8-6n+n^2)\right)\Big/\left(\sqrt{\left(75(-2+n)^2-40(7-8n+2n^2)+\frac{4(13-100n+185n^2-80n^3+10n^4)}{\left(35-840n+810n^2+500n^3-\frac{1125n^4}{2}+150n^5-\frac{25n^6}{2}+\frac{3}{2}\sqrt{3}\sqrt{-144-1200n+41260n^2+171520n^3-1146240n^4+2320800n^5-2491800n^6+1630800n^7-683925n^8+185000n^9-31250n^{10}+3000n^{11}-125n^{12}}\right)^{1/3}}+2\times2^{2/3}\left(70-1680n+1620n^2+1000n^3-1125n^4+300n^5-25n^6+3\sqrt{3}\sqrt{-144-1200n+41260n^2+171520n^3-1146240n^4+2320800n^5-2491800n^6+1630800n^7-683925n^8+185000n^9-31250n^{10}+3000n^{11}-125n^{12}}\right)^{1/3}\right)}\right)\Bigg)\Bigg),i\Bigg]$$

$$\text{Pochhammer}\Bigg[\frac{1}{60}\Bigg(90-15n-\sqrt{3}\sqrt{\left(75(-2+n)^2-40(7-8n+2n^2)+\frac{4(13-100n+185n^2-80n^3+10n^4)}{\left(35-840n+810n^2+500n^3-\frac{1125n^4}{2}+150n^5-\frac{25n^6}{2}+\frac{3}{2}\sqrt{3}\sqrt{-144-1200n+41260n^2+171520n^3-1146240n^4+2320800n^5-2491800n^6+1630800n^7-683925n^8+185000n^9-31250n^{10}+3000n^{11}-125n^{12}}\right)^{1/3}}+2\times2^{2/3}\left(70-1680n+1620n^2+1000n^3-1125n^4+300n^5-25n^6+3\sqrt{3}\sqrt{-144-1200n+41260n^2+171520n^3-1146240n^4+2320800n^5-2491800n^6+1630800n^7-683925n^8+185000n^9-31250n^{10}+3000n^{11}-125n^{12}}\right)^{1/3}\right)}+$$

$$\sqrt{6}\sqrt{\Bigg(75(-2+n)^2-40(7-8n+2n^2)-\frac{2(13-100n+185n^2-80n^3+10n^4)}{\left(35-840n+810n^2+500n^3-\frac{1125n^4}{2}+150n^5-\frac{25n^6}{2}+\frac{3}{2}\sqrt{3}\sqrt{-144-1200n+41260n^2+171520n^3-1146240n^4+2320800n^5-2491800n^6+1630800n^7-683925n^8+185000n^9-31250n^{10}+3000n^{11}-125n^{12}}\right)^{1/3}}-2^{2/3}\left(70-1680n+1620n^2+1000n^3-1125n^4+300n^5-25n^6+3\sqrt{3}\sqrt{-144-1200n+41260n^2+171520n^3-1146240n^4+2320800n^5-2491800n^6+1630800n^7-683925n^8+185000n^9-31250n^{10}+3000n^{11}-125n^{12}}\right)^{1/3}+}$$

$$\left(15\sqrt{3}\,n(8-6n+n^2)\right)\Big/\left(\sqrt{\left(75(-2+n)^2-40(7-8n+2n^2)+\frac{4(13-100n+185n^2-80n^3+10n^4)}{\left(35-840n+810n^2+500n^3-\frac{1125n^4}{2}+150n^5-\frac{25n^6}{2}+\frac{3}{2}\sqrt{3}\sqrt{-144-1200n+41260n^2+171520n^3-1146240n^4+2320800n^5-2491800n^6+1630800n^7-683925n^8+185000n^9-31250n^{10}+3000n^{11}-125n^{12}}\right)^{1/3}}+2\times2^{2/3}\left(70-1680n+1620n^2+1000n^3-1125n^4+300n^5-25n^6+3\sqrt{3}\sqrt{-144-1200n+41260n^2+171520n^3-1146240n^4+2320800n^5-2491800n^6+1630800n^7-683925n^8+185000n^9-31250n^{10}+3000n^{11}-125n^{12}}\right)^{1/3}\right)}\right)\Bigg)\Bigg),i\Bigg]\Bigg), \{i, 0, \text{Floor}\left[\frac{n}{5}\right]\}, \text{Method} \to \text{HypergeometricTermPFQ}\Bigg]$$

future work

counting successive vertex orderings for nice graphs (e.g., hypercube Q_n), and nice classes of graphs (e.g., strongly regular / distance-transitive graphs)

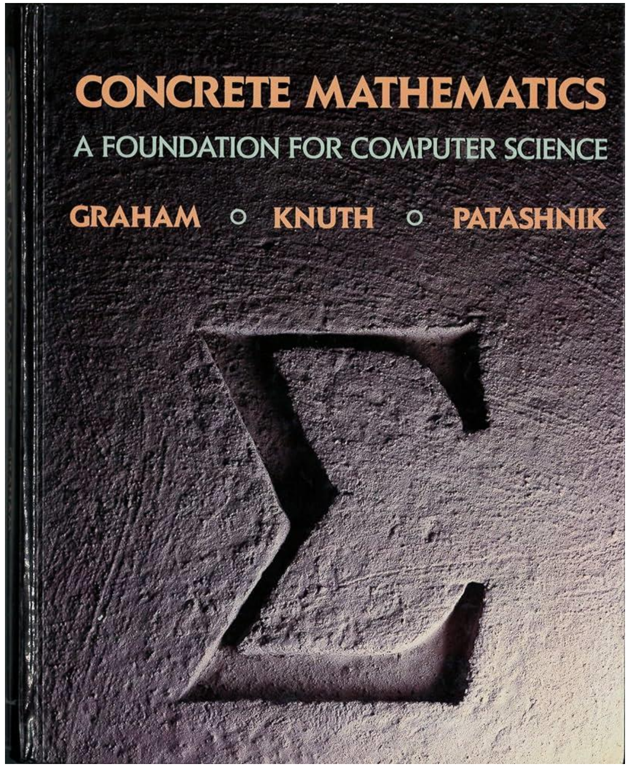
recently, T. Amdberhan proved that $\sum_{i=0}^{\lfloor n/3 \rfloor} \prod_{j=1}^{i} \frac{-\binom{n-3j}{3}}{\binom{n}{3}-\binom{n-3j}{3}} = \frac{n}{3} \sum_{i=0}^{\lfloor n/3 \rfloor} \prod_{j=1}^{i} \frac{-\binom{n+3-1-3j}{3}}{\binom{n}{3}-\binom{n-3j}{3}}$ (in fact, this generalizes from 3 to t); together with the Pfaff–Saalschütz identity, this gives an alternate proof of the $L(K_n^{(3)})$ theorem avoiding Sheppard's theorem and Gosper's algorithm. does a similar method apply for proving the $L(K_{m,n}^{(1,2)})$ theorem?

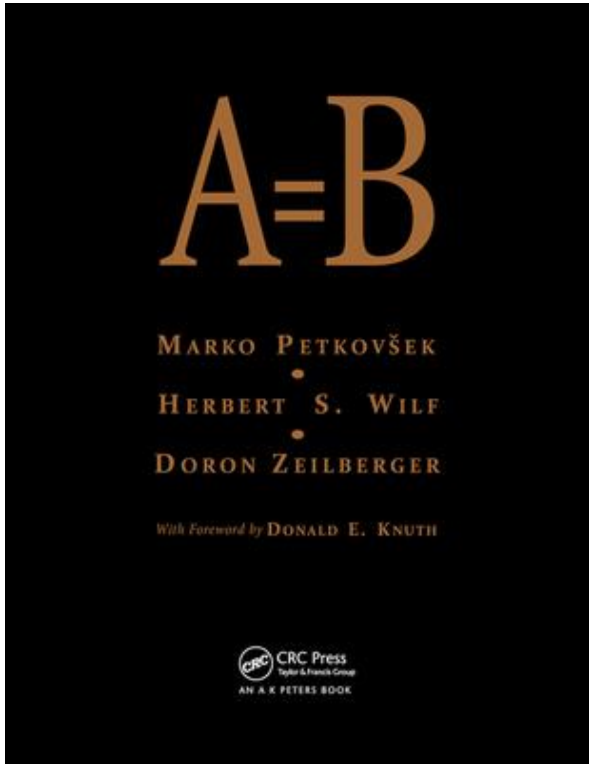
how do we apply hypergeometric techniques to the $K_{m,n,p}^{(1,1,1)}$ case?

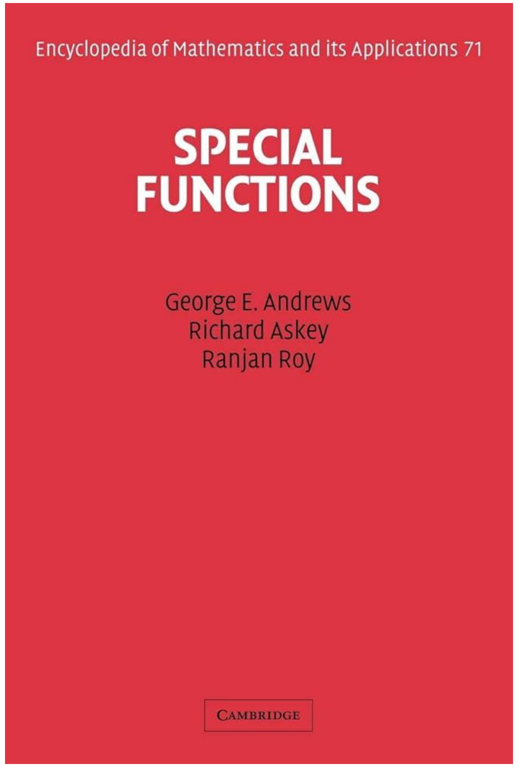
are there combinatorial arguments for the product formulas like Stanley's for $L(K_n)$?

algorithms for computing $\sigma(G)$ for symmetric graphs / general graphs

further reading







Thank you for listening!