# Kahn's basis conjecture is false

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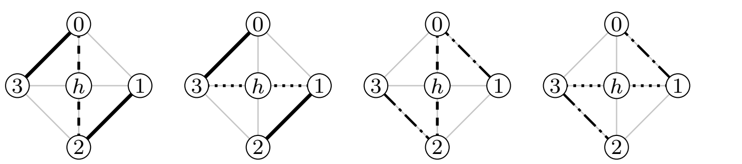
Jeff Kahn conjectured in 1991 that, given an $n \times n$ array of bases $(B_{ij})$ of an $n$-dimensional vector space $V$, one can choose an element $v_{ij}$ from each $B_{ij}$ such that the chosen elements in every row and in every column are again bases. More generally, we may replace $V$ with a matroid of rank $n$.

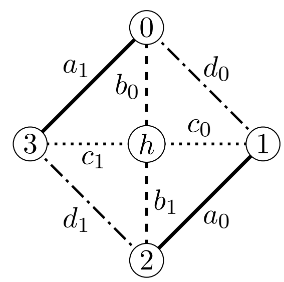
We construct a counterexample (discovered by GPT-5.6 Pro) for $n = 4$, which comes from the cycle matroid of a wheel.

We use the four spanning trees

$$X = \{a_0, a_1, b_0, b_1\}, \quad Y = \{a_0, a_1, c_0, c_1\},$$
$$Z = \{b_0, b_1, d_0, d_1\}, \quad W = \{c_0, c_1, d_0, d_1\}.$$




$X$ $Y$ $Z$ $W$




The wheel graph $W_4$

Put these four bases into the array

$$\mathcal{A} = \begin{pmatrix} X & X & X & Y \\ X & X & X & Y \\ X & X & X & Y \\ Z & Z & Z & W \end{pmatrix}.$$

We will show that no choice of one edge from each of the 16 cells can make every row and every column a spanning tree.

From the wheel, one reads off the following completion rules. If a top row or a left column omits one edge from its three $X$-entries, then its boundary entry can be chosen exactly as follows:

| omitted from $X$ | choice on the right, from $Y$ | choice on the bottom, from $Z$ |
|---|---|---|
| $a_i$ | $a_i$ or $c_i$ | $d_i$ |
| $b_i$ | $c_{\bar{\imath}}$ | $b_i,\ d_0,\ d_1$ |

(1)

At the bottom-right corner, we need only the following two rules:

$$\begin{aligned} &Y \setminus \{a_i\} \text{ can be completed by an edge of } W \text{ only with } d_{\bar{\imath}}, \\ &Z \setminus \{b_i\} \text{ can be completed by an edge of } W \text{ only with } c_i. \end{aligned} \tag{2}$$

Here $\bar{\imath} = 1 - i$ for $i \in \{0, 1\}$.

**Claim.** *The array $\mathcal{A}$ has no successful choice.*

*Proof.* Assume the contrary. In each of the first three rows, the three choices from the left $3 \times 3$ block come from $X$ and must be distinct, so exactly one edge of $X$ is omitted from that row. The same holds for each of the first three columns. Counting how often each edge of $X$ appears in the $3 \times 3$ block shows that the three row omissions and the three column omissions are the same multiset.

No edge can appear twice in this multiset. Indeed, if two top rows omitted $b_i$, then by (1) the rightmost choices in those two rows would both have to be $c_{\bar{\imath}}$, so the last column could not be a four-element basis. Likewise, if two left columns omitted $a_i$, then the bottom choices in those two columns would both have to be $d_i$, so the last row could not be a four-element basis. Thus each edge of $X$ is omitted equally often by rows and by columns. A repeated $b_i$ is impossible when viewed among the row omissions, while a repeated $a_i$ is impossible when viewed among the column omissions. Hence the three omissions are distinct.

For some $i \in \{0, 1\}$, they therefore have one of the following two forms:

$$\{a_0, a_1, b_i\} \qquad \text{or} \qquad \{a_i, b_0, b_1\}.$$

*First suppose that the omissions are* $\{a_0, a_1, b_i\}$*.* Along the bottom row, the omissions $a_0$ and $a_1$ force the choices $d_0$ and $d_1$. The choice corresponding to the omission $b_i$ must therefore be $b_i$, since its only other possibilities are $d_0$ and $d_1$, which are already present. Thus the first three bottom entries are

$$Z \setminus \{b_{\bar{\imath}}\}.$$

By (2), the bottom-right entry must be $c_{\bar{\imath}}$.

But the top row whose omission is $b_i$ already has $c_{\bar{\imath}}$ as its rightmost entry, by (1). Hence the rightmost column contains $c_{\bar{\imath}}$ twice and cannot be a spanning tree. This is a contradiction.

*Now suppose that the omissions are* $\{a_i, b_0, b_1\}$*.* Along the rightmost column, the omissions $b_0$ and $b_1$ force the choices $c_1$ and $c_0$. The choice corresponding to the omission $a_i$ must therefore be $a_i$, since its only other possibility, $c_i$, is already present. Thus the first three rightmost entries are

$$Y \setminus \{a_{\bar{\imath}}\}.$$

By (2), the bottom-right entry must be $d_i$.

But the bottom entry in the column whose omission is $a_i$ is already $d_i$, by (1). Hence the bottom row contains $d_i$ twice and cannot be a spanning tree. This is again a contradiction.

Both cases are impossible, so $\mathcal{A}$ is a counterexample to Kahn's basis conjecture for $n = 4$. Since the cycle matroid of a graph is represented over every field by a reduced signed incidence matrix, the same array is also a counterexample to the vector-space version of the conjecture.

Similar constructions exist for all $n \geq 4$; see `https://boonsuan.github.io/kahn.pdf` for more information.