Some solutions to Yufei Zhao's book "Graph Theory and Additive Combinatorics"

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I have done my best to indicate my sources whenever the solution provided below is not my own. Nevertheless, nothing in this document should be taken as original. I thank my friends for their comments and assistance in preparing this document.

If you have any comments or bugs to report, you may reach me at `hbs@u.nus.edu`.

The latest version of this document can be found at `https://boonsuan.github.io/gtac_solutions.pdf`.

0. Appetizer: Triangles and Equations

0.1. Schur's Theorem

Exercise 0.1.5. We first induct on r to prove that $N_r = 1 + r!\sum_{i=0}^{r} \frac{1}{i!}$. We have

$$N_1 = 3 = 1 + 1!\Big(\frac{1}{0!} + \frac{1}{1!}\Big),$$

and for the inductive step with $r \geq 2$ we have

$$\begin{aligned} N_r &= r(N_{r-1} - 1) + 2 \\ &= r\Big(1 + (r-1)!\sum_{i=0}^{r-1}\frac{1}{i!} - 1\Big) + 2 \\ &= r!\sum_{i=0}^{r-1}\frac{1}{i!} + 2 \\ &= 1 + r!\sum_{i=0}^{r}\frac{1}{i!}. \end{aligned}$$

To prove that $N_r = \lceil r!e \rceil$, we must show that $N_r - 1 < r!e \leq N_r$. The first inequality amounts to

$$r!\sum_{i=0}^{r}\frac{1}{i!} < r!e,$$

which is clear from the fact that $e = \sum_{i=0}^{\infty}\frac{1}{i!}$. The second inequality follows from the fact that

$$\begin{aligned} \sum_{i=r+1}^{\infty}\frac{1}{i!} &= \frac{1}{r!}\Big(\frac{1}{r+1} + \frac{1}{(r+1)(r+2)} + \cdots\Big) \\ &\leq \frac{1}{r!}\Big(\frac{1}{r+1} + \frac{1}{(r+1)^2} + \cdots\Big) = \frac{1}{rr!} \leq \frac{1}{r!}. \end{aligned}$$

Exercise 0.1.8. It is convenient to introduce the *multicolor Ramsey number* $R_r(k_1, \ldots, k_r)$, defined as the smallest N such that any r-edge-coloring of K_N has a k_i-clique whose edges are all of color i, for some i. We are then asked to prove that $R_r(k, \ldots, k)$ is finite for all k and r.

This is a standard argument; see for example `https://en.wikipedia.org/wiki/Ramsey%27s_theorem#Proof`.

We first prove by induction on $k + l$ that $R_2(k, l)$ is finite for all k and l. The base case is the fact that $R_2(k, 2) = R_2(2, k) = k$. For the inductive step, we show that $R_2(k, l) \leq R_2(k-1, l) + R_2(k, l-1)$. Indeed, if we edge color an $(m+n)$-clique with the colors red and blue (where $m = R_2(k-1, l)$ and $n = R_2(k, l-1)$) and fix a vertex v, the pigeonhole principle guarantees that there are at least m red edges or n blue edges adjacent to v. In the first case we can take m vertices joined to v by red edges to obtain either a blue l-clique in which case we are done, or a red $(k-1)$-clique which then forms a k-clique when combined with the red edges from v. The second case works analogously.

Now we prove that

$$R_r(k_1,\dots,k_r) \le R_{r-1}\big(k_1,\dots,k_{r-2},R_2(k_{r-1},k_r)\big) =: N;$$

induction on r then completes the proof. Suppose we are given an r-edge-coloring of K_N. Treating colors $r-1$ and r as the same color yields an $(r-1)$-edge-coloring of K_N, and the definition of N then yields a k_i-clique of color i for some $1 \le i \le r-2$ in which case we are done, or a $R_2(k_{r-1},k_r)$-clique of the combined color $\{r-1,r\}$, which yields in the original coloring a k_{r-1}-clique of color $r-1$ or a k_r-clique of color r.

Exercise 0.1.10. We write $R_r^{(s)}(k_1,\dots,k_r)$ for the *hypergraph Ramsey number*, which is the minimum number n such that any r-coloring of the s-uniform hypergraph on n vertices contains a clique of size k_i whose hyperedges all have color i for some $1 \le i \le r$.

Adapted from Jacob Fox's notes: `https://math.mit.edu/~fox/MAT307-lecture06.pdf`.

We first consider the case with $r=2$ colors, where we use the interpretation of $R_2^{(s)}(k,l)$ as being the minimum n such that any s-uniform hypergraph on n vertices contains an independent set of size k or a clique of size l. We induct on the uniformity s to prove that

$$R_2^{(s)}(k,l) \le R_2^{(s-1)}\Big(R_2^{(s)}(k-1,l),R_2^{(s)}(k,l-1)\Big)+1 =: N$$

for all $k,l \ge s \ge 1$. For the base case, we observe that $R_2^{(2)}(k,l)$ is finite for all k and l by the standard Graph Ramsey Theorem, and that $R_2^{(s)}(k,s) = R_2^{(s)}(s,k) = k$ for all $k \ge s$. For the inductive step, fix s, k, and l, and assume that the quantities $R_2^{(s)}(k-1,l)$, $R_2^{(s)}(k,l-1)$, and $R_2^{(s-1)}(u,v)$ for all u, v are all finite. Given an s-uniform hypergraph H on N vertices, fix $v \in V(H)$ and define the *link of* v to be the $(s-1)$-uniform hypergraph $L(v)$ on $V(H)\setminus\{v\}$ whose hyperedges are $(s-1)$-element sets A such that $A\cup\{v\} \in E(H)$. (This generalizes the idea of the neighborhood of a graph vertex.) The definition of N then implies that $L(v)$ has an independent set K of size $R_2^{(s)}(k-1,l)$ or a clique L of size $R_2^{(s)}(k,l-1)$. In the first case, $R'\cup\{v\}$ is not a hyperedge of H whenever $R' \in \binom{K}{s-1}$, and so applying the inductive hypothesis to the induced hypergraph $H[K]$ then yields either an independent set K_1 of size $k-1$ in which case $K_1\cup\{v\}$ is an independent set of size k in H, or a clique L_1 of size l. Similarly for the second case, $R'\cup\{v\}$ is a hyperedge of H whenever $R' \in \binom{L}{s-1}$, and considering $H[L]$ yields either an independent set K_2 of size k or a clique L_2 of size $l-1$ that gives rise to the l-clique $L_2\cup\{v\}$ in H.

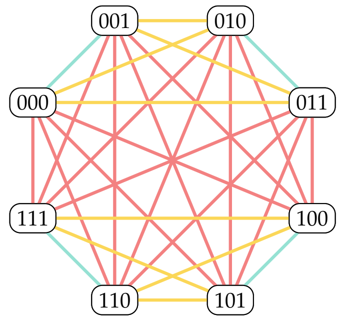
Finally, for the case of $r>2$ colors, the combined color argument from exercise 0.1.8 yields

$$R_r^{(s)}(k_1,\dots,k_r) \le R_{r-1}^{(s)}\Big(k_1,\dots,k_{r-2},R_2^{(s)}(k_{r-1},k_r)\Big),$$

from which the general result follows by induction on r.

Remark (Erdős–Szekeres convex polygon theorem). The hypergraph Ramsey theorem implies that for any $m \geq 4$, there exists n such that given any configuration of n points in the plane with no three on a line, we can find m points that form a convex polygon. The idea is to set $n := R_2^{(4)}(5, m)$ and to define a 4-uniform hypergraph on the set of n points comprising the sets of four points that form convex quadrilaterals; then the hypergraph Ramsey theorem yields either an independent set of five points, which is impossible (why?), or a clique of size m, which can be shown to be our desired set of m points. □

Remark (Proposition 0.1.12). *An r-edge-coloring of K_{2^r} avoiding monochromatic triangles:* Consider the r-edge-coloring of the complete graph K_{2^r} on vertex set $\{0,1\}^r$, defined by coloring each edge with the smallest index at which the vertices differ. Then, given three vertices x, y, z, if xy and yz are colored the same, meaning that they both differ at index i, then x and z agree at index i; and so the edge xz must be colored differently from xy and yz.



Interestingly, this construction gives a decomposition of K_{2^r} into complete bipartite graphs: For example, $K_8 = K_{4,4} \cup 2K_{2,2} \cup 4K_{1,1}$, as can be seen on the right. □

Exercise 0.1.14. Given an $(r-1)$-coloring $c\colon [N(r-1)-1] \to [r-1]$ avoiding monochromatic solutions to $x + y = z$, we may define a coloring $\widetilde{c}\colon [3N(r-1)-2] \to [r]$ by

$$\widetilde{c}(n) := \begin{cases} c(n), & \text{if } n < N(r-1); \\ r, & \text{if } N(r-1) \leq n < 2N(r-1); \\ c(n - 2N(r-1) + 1), & \text{if } 2N(r-1) \leq n. \end{cases}$$

Clearly there are no solutions to $x + y = z$ that are of color r, and so any monochromatic solution would have to satisfy $x < N(r-1)$ and $2N(r-1) \leq y, z$. But then

$$x + \big(y - 2N(r-1) + 1\big) = z - 2N(r-1) + 1$$

would be a monochromatic triple in c! Thus $\widetilde{c}$ avoids monochromatic solutions to $x + y = z$, and we have $N(r) \geq 3N(r-1) - 1$ as needed.

It is easy to verify that $N(1) = 2$ and $N(2) = 5$; these both satisfy $N(r) \geq (3^r + 1)/2$. Inductively, we then have

$$N(r) \geq 3N(r-1) - 1 \geq 3(3^{r-1} + 1)/2 - 1 = (3^r + 1)/2.$$

Finally, there exists an r-coloring of $[(3^r - 1)/2]$ avoiding monochromatic solutions to $x + y = z$, and we may transfer it to an edge-coloring of $K_{(3^r+1)/2}$ as in the proof of Schur's theorem; any monochromatic triangles in this complete graph would give us monochromatic solutions to $x + y = z$, and so we conclude that the graph does not contain any monochromatic triangles.

Exercise 0.1.15. In exercise 0.1.8, we proved that

$$R_2(s,t) \le R_2(s-1,t) + R_2(s,t-1).$$

Since $R_2(s,2) = s = \binom{s+2-2}{s-1}$ and $R_2(2,t) = t = \binom{2+t-2}{2-1}$, the desired inequality holds at the boundaries, and induction together with the identity $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ then gives the claim.

Exercise 0.1.16. (a) True. This follows from Goodman's formula, which states that the number Δ of monochromatic triangles in a red-blue edge coloring of the complete graph K_n is given by

Adapted from David Conlon's notes: `http://www.its.caltech.edu/~dconlon/RamseyLecture1.pdf`.

$$\Delta = \frac{1}{2}\left[\sum_v \binom{d_G(v)}{2} + \sum_v \binom{d_{\overline{G}}(v)}{2} - \binom{n}{3}\right],$$

where G and $\overline{G}$ denote respectively the graphs comprising the red and blue edges in the coloring. We can then minimize Δ by taking $d_G(v) = (n-1)/2$, which yields $\Delta \ge \frac{1}{24}n(n-1)(n-5) = \frac{1}{4}\binom{n}{3} + O(n^2)$.

(b) False. See J. Cummings, D. Král, F. Pfender, K. Sperfeld, A. Treglown, and M. Young, *J. Combinatorial Theory* **B103** (2013), 489–503, where the minimum fraction of monochromatic triangles is shown to be $1/25$ asymptotically.

(c*) False. Although conjectured to be true by Erdős in the 1960s, Andrew Thomason constructed difficult counterexamples in *Journal of the London Mathematical Society* (2) **39** (1989), 246–255. He later presented simpler constructions involving graph tensor products in *Combinatorica* **17** (1997), 125–134.

0.2. Progressions

Exercise 0.2.2. Color 0 blue, $\{1,2\}$ red, $\{3,4,5\}$ blue, $\{6,7,8,9\}$ red, etc. If $a < 0$, give it the same color as $-a$. Since the monochromatic blocks eventually become longer than any fixed common difference d, any infinite arithmetic progression will be unable to 'jump over' such a block completely, and so it cannot be monochromatic.

Remark (Recent improvements for bounds on Szemerédi's theorem). In 2023, Zander Kelley and Raghu Meka [arXiv:2302.05537 [math.NT] (2023), 79 pages] proved that 3-AP-free subsets of $[N]$ have size at most

$$N \exp(-c(\log N)^{1/12})$$

for some $c > 0$; the exponent $1/12$ was quickly improved to $1/9$ by Thomas Bloom and Olof Sisask in the same year [arXiv:2309.02353 [math.NT] (2023), 9 pages]. Thus Behrend's bound is close to the truth, as long suspected.

For $k \ge 5$, James Leng, Ashwin Sah, and Mehtaab Sawhney have recently obtained the improved bound $CN\exp\big(-(\log\log N)^{c_k}\big)$; see arXiv:2402.17995 [math.CO] (2024), 13 pages. □

1. Forbidding a Subgraph

1.1. Forbidding a Triangle: Mantel's Theorem

Exercise 1.1.3. We begin with a lemma: If $x, y, z \in \mathbf{R}^d$ satisfy $|x|, |y|, |z| \geq 1$, then we cannot have $|x+y|, |x+z|, |y+z| < 1$. It suffices to observe that

Adapted from a post of Misha Lavrov: `https://math.stackexchange.com/a/4370689/`. I thank Clarence Chew for the proof of the lemma.

$$\begin{aligned}
&|x+y|^2 + |x+z|^2 + |y+z|^2 \\
&\quad = |x|^2 + 2\,x \cdot y + |y|^2 \\
&\qquad + |x|^2 + 2\,x \cdot z + |z|^2 \\
&\qquad + |y|^2 + 2\,y \cdot z + |z|^2 \\
&\quad = |x+y+z|^2 + |x|^2 + |y|^2 + |z|^2 \geq 3.
\end{aligned}$$

Consider i.i.d. $X_1, \ldots, X_n$ drawn from the same distribution as X. The expected number of sets $\{X_i, X_j\}$ with $i \neq j$ and $|X_i + X_j| \geq 1$ is $M := \binom{n}{2} \Pr(|X+Y| \geq 1)$, whereas the expected number of X_i with $|X_i| \geq 1$ is $N := n \Pr(|X| \geq 1)$.

Say $|X_1|, \ldots, |X_k| \geq 1$. Define a graph G on $[k]$ where ij is an edge iff $|X_i + X_j| < 1$. Then G is triangle-free by the lemma above, and so Mantel's theorem implies that there are at most $k^2/4$ sets $\{i, j\}$ with $|X_i + X_j| < 1$. Thus in expectation we have $M \geq N^2/4$; sending $n \to \infty$ then completes the proof.

Exercise 1.1.4. A triangle xyz arises from an edge xy together with a common neighbor z of x and y. Consider an edge xy. The vertices x and y have at least $\deg(x) + \deg(y) - n$ common neighbors, and thus we have by Cauchy–Schwarz

$$\begin{aligned}
\sum_{xy \in E(G)} \Big(\deg(x) + \deg(y) - n\Big) &= \sum_{x \in V(G)} \deg(x)^2 - nm \\
&\geq \frac{1}{n}\Big(\sum_{x \in V(G)} \deg(x)\Big)^2 - nm \\
&= \frac{4m^2}{n} - nm.
\end{aligned}$$

Since every triangle has three edges, we see that our graph has at least

$$\frac{1}{3}\Big(\frac{4m^2}{n} - nm\Big) = \frac{4m}{3n}\Big(m - \frac{n^2}{4}\Big)$$

triangles as needed.

Exercise 1.1.5. Let G be an n-vertex nonbipartite triangle-free graph, and consider a smallest cycle C, so that $3 < |C| =: k \leq n$. Let us count the number of edges in G. Minimality implies that every vertex of $A := G \setminus C$ can be connected to at most two vertices of C, so there are at most $2(n-k)$ edges between A and C. Mantel's theorem gives at most $(n-k)^2/4$ edges within A, and there are k edges within C by minimality again, so adding these up and using the fact that $k \geq 5$ yields the desired bound.

See Lemma 1 in P. Erdös, *Illinois Journal of Mathematics* **6** (1962), 122–127.

Exercise 1.1.6. Let $v \in G$ be a vertex of maximum degree, and write N_v for its neighborhood, which is an independent set as G is triangle-free. We compute

$$\begin{aligned}\left\lfloor \frac{n^2}{4} \right\rfloor &\geq \deg(v)\Big(n - \deg(v)\Big) \\ &\geq \sum_{x \notin N_v} \deg(x) \\ &= 2e(G[V \setminus N_v]) + e(N_v, V \setminus N_v) \\ &= e(G) + e(G[V \setminus N_v]) \\ &\geq \left\lfloor \frac{n^2}{4} \right\rfloor - k + e(G[V \setminus N_v]),\end{aligned}$$

from which the result follows, since removing the edges of $G[V \setminus N_v]$ makes G bipartite.

Adapted from a post of Lanchao Wang: `https://math.stackexchange.com/a/4914839/`.

Exercise 1.1.7. Suppose for contradiction that G is not bipartite, and let C be a smallest cycle, so that it has odd length $k \geq 5$. Then C is chordless, and so there are more than $k(2n/5 - 2) \geq 2(n-k)$ edges between C and $A := G \setminus C$. Since $|A| = n - k$, the pigeonhole principle admits $v \in A$ that is adjacent to at least three vertices of C, but this gives rise to a smaller odd cycle, contradicting the minimality of C.

Adapted from a post of Alex Ravsky: `https://math.stackexchange.com/a/2423710/`

Exercise 1.1.8*. We induct on n. Suppose inductively that a graph on $n-2$ vertices with at least $\lfloor (n-2)^2/4 \rfloor + 1$ edges contains at least $\lfloor (n-2)/2 \rfloor$ triangles, and let G be an n-vertex graph with $\lfloor n^2/4 \rfloor + 1 = \lfloor (n-2)^2/4 \rfloor + 1 + (n-1)$ edges.

Suppose for contradiction that G has less than $\lfloor n/2 \rfloor$ triangles. Then we can find an edge xy of G such that x and y have no common neighbors, since $3(\lfloor n/2 \rfloor - 1) < \lfloor n^2/4 \rfloor + 1$. Thus $\deg(x) + \deg(y) \leq n$. Let H be the subgraph of G with vertices x and y removed. Then H is an $(n-2)$-vertex graph with at least $\lfloor (n-2)^2/4 \rfloor + 1$ edges, and so it contains at least $\lfloor (n-2)/2 \rfloor$ triangles. But there are at most $\lfloor (n-2)^2/4 \rfloor$ edges between $N(x) \setminus y$ and $N(y) \setminus x$ in H, so there is an edge contained completely in one of these sets. This gives rise to a triangle in G, which completes the induction.

Adapted from David Conlon's notes: `https://www.its.caltech.edu/~dconlon/EGTSheet1Sol.pdf`

Exercise 1.1.9*. To do...

Exercise 1.1.10*. To do...

See P. Erdös, *Illinois J. Mathematics* **6** (1962), 122–127.

1.2. *Forbidding a Clique: Turán's Theorem*

Exercise 1.2.5. Writing $k = n \bmod r$, we see that $T_{n,r}$ has k parts of size $\lceil n/r \rceil$ and $r - k$ parts of size $\lfloor n/r \rfloor$, so that

$$e(T_{n,r}) = \binom{k}{2} \left\lceil \frac{n}{r} \right\rceil^2 + k(r-k) \left\lceil \frac{n}{r} \right\rceil \left\lfloor \frac{n}{r} \right\rfloor + \binom{r-k}{2} \left\lfloor \frac{n}{r} \right\rfloor^2.$$

Proving the bound then amounts to optimizing the function

$$(x_1, \ldots, x_r) \mapsto \sum_{1 \leq i < j \leq r} x_i x_j$$

subject to the constraint $x_1 + \cdots + x_r = n$, and it can be shown that the maximum is achieved when $x_1 = \cdots = x_r$.

Remark (Fourth proof of Turán's theorem). The inequality

$$\sum_{v \in V} \frac{1}{n - \deg v} \geq \frac{n}{n - (\sum_{v \in V} \deg v)/n}$$

follows from the convexity of $x \mapsto 1/(n-x)$ on $(0, n)$; in particular, we use Jensen's inequality in the form

$$\frac{1}{n} \sum_i f(x_i) \geq f\Big(\frac{1}{n} \sum_i x_i\Big).$$

Exercise 1.2.8. To do...

Exercise 1.2.9*. To do...

Exercise 1.2.10. To do...

1.3. Turán Density and Supersaturation

Remark (Proposition 1.3.1). In the following computation, $S \subset V(G)$ will be a set of n vertices from a graph G on $n+1$ vertices; we will write v_S for the single vertex of G that does not belong to S. We have

$$\begin{aligned}
\mathbb{E}_{\substack{S \subset V(G) \\ |S| = n}} \frac{e(G[S])}{\binom{n}{2}} &= \frac{1}{\binom{n+1}{n}} \sum_{\substack{S \subset V(G) \\ |S| = n}} \frac{e(G[S])}{\binom{n}{2}} \\
&= \frac{1}{\binom{n+1}{2}(n-1)} \sum_{\substack{S \subset V(G) \\ |S| = n}} (e(G) - \deg(v_S)) \\
&= \frac{1}{\binom{n+1}{2}(n-1)} \Big((n+1)e(G) - 2e(G)\Big) \\
&= \frac{e(G)}{\binom{n+1}{2}}.
\end{aligned}$$

□

Remark (Some details in Theorem 1.3.4). One way to understand the equivalent statement of the theorem is to take $\epsilon_n = 1/n$ and obtain corresponding δ_n from the theorem so that every sufficiently large graph with at least $(\pi(H) + 1/n)\binom{n}{2}$ edges contains at least $\delta_n n^{v(H)}$ copies of H as a subgraph; having $o(n^{v(H)})$ copies of H then means that we eventually have less than $\delta_n n^{v(H)}$ copies of H for each n, so that the edge density is at most $\pi(H) + o(1)$.

To show that $\mathbb{E}\, X = e(G)/\binom{n}{2}$, we compute

$$\begin{aligned}
\mathbb{E}_{S \in \binom{V(G)}{n_0}} \frac{e(G[S])}{\binom{n_0}{2}} &= \frac{1}{\binom{n}{n_0}} \sum_{S \in \binom{V(G)}{n_0}} \frac{e(G[S])}{\binom{n_0}{2}} \\
&= \frac{1}{\binom{n}{2}\binom{n-2}{n_0-2}} \sum_{S \in \binom{V(G)}{n_0}} \frac{e(G[S])}{\binom{n_0}{2}}.
\end{aligned}$$

Thus it suffices to show that

$$\binom{n-2}{n_0-2}e(G) = \sum_{S\in\binom{V(G)}{n_0}} \frac{e(G[S])}{\binom{n_0}{2}},$$

but this simply states that every edge xy of G is counted in the sum $\binom{n-2}{n_0-2}$ times, which holds because that is the number of ways to choose the set S after forcing it to include the vertices x and y.

To see that $\Pr(X \geq \pi(H) + \epsilon/2) \geq \epsilon/2$, observe that

$$\int_0^1 \Pr(X \geq t)\, dt = \mathbb{E}\, X \geq \pi(H) + \epsilon,$$

so $\Pr(X \geq \pi(H) + \epsilon/2) < \epsilon/2$ would imply that

$$\begin{aligned}\mathbb{E}\, X &= \int_0^{\pi(H)+\epsilon/2} \Pr(X \geq t)\, dt + \int_{\pi(H)+\epsilon/2}^1 \Pr(X \geq t)\, dt \\ &\leq \pi(H) + \frac{\epsilon}{2} + \frac{\epsilon}{2}\Big(1 - \pi(H) - \frac{\epsilon}{2}\Big) \\ &< \pi(H) + \epsilon.\end{aligned}$$

□

Exercise 1.3.5. To do...

Exercise 1.3.6. To do...

Exercise 1.3.7. To do...

1.4. Forbidding a Complete Bipartite Graph: Kővári–Sós–Turán Theorem

Exercise 1.4.11. Let the graph G have m edges, and parts A and B of sizes a and b respectively. We bound $e(G)$ by double counting the number X of copies of $K_{2,1}$ in G *with the right vertex in* B. The upper bound is $X \leq \binom{a}{2}$ since G is C_4-free. The lower bound is given by

$$X = \sum_{v\in V(B)} \binom{\deg v}{2} = \sum_{v\in V(B)} f_2(\deg v) \geq b f_2\Big(\frac{m}{b}\Big),$$

where we have used the convexity of $f_2(x) := x(x-1)/2[x \geq 1]$. Since we can assume $m \geq b$ (by considering say the bipartite graph between A and B with a fixed vertex $a \in A$ and every vertex of B adjacent to a), we can put the lower and upper bounds together to get

$$\frac{b}{2}\cdot\frac{m}{b}\Big(\frac{m}{b} - 1\Big) - \binom{a}{2} \leq 0.$$

The left-hand side is quadratic in m, and evaluating it at $m = ab^{1/2} + b$ gives $\frac{1}{2}a(b^{1/2}+1) > 0$, so we conclude that $m \leq ab^{1/2} + b$.

Exercise 1.4.12. To do...

Exercise 1.4.13. To do...

Exercise 1.4.14. To do...

1.5. Forbidding a General Subgraph: Erdős–Stone–Simonovits Theorem

Exercise 1.5.8. The argument is a straightforward modification of the proof that $\mathrm{ex}(n, K^{(3)}_{s,s,s}) \lesssim_s n^{3-1/s^2}$ (Theorem 1.5.7). We nonetheless write it out carefully as practice.

Let G be a $K^{(3)}_{r,s,t}$-free 3-graph with n vertices and m edges. Let X denote the number of copies of $K^{(3)}_{1,s,1}$ in G. (When $s = 1$, we count each copy three times.)

Upper bound on X. Given a set S of s vertices, consider the set A of all unordered pairs of distinct vertices that would form a $K^{(3)}_{1,s,1}$ with S (i.e., every triple formed by combining a pair in A and a vertex in S is an edge of G). Note that A is the edge set of a graph on the same n vertices. If A contains a $K_{r,t}$, then together with S we would have a $K^{(3)}_{r,s,t}$. Thus, A is $K_{r,t}$-free, and hence by Theorem 1.4.2, $|T| = O_{r,t}(n^{2-1/r})$. Therefore,

$$X \lesssim_{r,t} \binom{n}{s} n^{2-1/r} \lesssim_{r,s,t} n^{s+2-1/r}.$$

Lower bound on X. We write $\deg(u, v)$ for the number of edges in G containing both u and v. Then, summing over all unordered pairs of distinct vertices u, v in G, we have

$$X = \sum_{u,v} \binom{\deg(u,v)}{s}.$$

As in the proof of Theorem 1.4.2, let

$$f_s(x) = \begin{cases} x(x-1)\dots(x-s+1)/s! & \text{if } x \geq s-1; \\ 0 & \text{if } x < s-1. \end{cases}$$

Then, f_s is convex and $f_s(x) = \binom{x}{s}$ for all nonnegative integers x. Since the average of $\deg(u, v)$ is $3m/\binom{n}{2}$, we have

$$X = \sum_{u,v} f_s(\deg(u,v)) \geq \binom{n}{2} f_s\left(\frac{3m}{\binom{n}{2}}\right).$$

Combining the upper and lower bounds, we have

$$\binom{n}{2}\left(\frac{3m}{\binom{n}{2}}\right)^s \lesssim_{r,s,t} n^{s+2-1/r}$$

and hence

$$m = O_{r,s,t}(n^{3-1/(rs)}).$$

Exercise 1.5.10. To do...

Exercise 1.5.11. To do...

1.6. Forbidding a Cycle

Exercise 1.6.8. Given a tree T with k edges and an n-vertex graph G with kn edges, Lemma 1.6.7 yields a subgraph H with minimum degree at least $k+1$. Ordering the vertices $v_1, \dots, v_{k+1}$ of the tree T so that each induced subgraph $G[v_1, \dots, v_l]$ is connected, we may embed T in H one edge at a time, and there will always be enough unused edges at each vertex due to the minimum degree condition.

Remark (The extremal number of trees). Erdős and Sós conjectured in 1962 that every n-vertex graph with more than $n(k-2)/2$ edges must contain every k-vertex tree for $n \geq k$. A proof for large k was announced in the 1990s by M. Ajtai, J. Komlós, M. Simonovits, and E. Szemerédi, but thirty years have passed and it has yet to appear.

See however these slides from a 2015 talk of Simonovits that give an outline of their approach: `https://imada.sdu.dk/Research/GT2015/Talks/Slides/simonovits.pdf`.

1.7. Forbidding a Sparse Bipartite Graph: Dependent Random Choice

Exercise 1.7.7. To do. . .

Exercise 1.7.8. To do. . .

Exercise 1.7.9. To do. . .

1.8. Lower Bound Constructions: Overview

(This section has no exercises.)

1.9. Randomized Constructions

Exercise 1.9.5. To do. . .

Exercise 1.9.6. To do. . .

1.10. Algebraic Constructions

(This section has no exercises.)

1.11. Randomized Algebraic Constructions

(This section has no exercises.)

2. *Graph Regularity Method*

2.1. *Szemerédi's Graph Regularity Lemma*

Exercise 2.1.4. To do...

Exercise 2.1.5. To do...

Exercise 2.1.6. To do...

Exercise 2.1.22. To do...

Exercise 2.1.23. To do...

Exercise 2.1.24. To do...

Exercise 2.1.25. To do...

Exercise 2.1.27. To do...

Exercise 2.1.28*. To do...

2.2. *Triangle Counting Lemma*

(This section has no exercises.)

2.3. *Triangle Removal Lemma*

Exercise 2.3.6. To do...

2.4. *Graph Theoretic Proof of Roth's Theorem*

Exercise 2.4.6*. To do...

2.5. *Large 3-AP-Free Sets: Behrend's Construction*

Exercise 2.5.4. To do...

Exercise 2.5.5*. To do...

2.6. *Graph Counting and Removal Lemmas*

Exercise 2.6.6. To do...

2.7. *Exercises on Applying Graph Regularity*

Exercise 2.7.1. To do...

Exercise 2.7.2. To do...

Exercise 2.7.3. To do...

Exercise 2.7.4. To do...

Exercise 2.7.5*. To do...

Exercise 2.7.6*. To do...

2.8. Induced Graph Removal and Strong Regularity

Exercise 2.8.8. To do...

2.9. Graph Property Testing

(This section has no exercises.)

2.10. Hypergraph Removal and Szemerédi's Theorem

Exercise 2.10.3. To do...

Exercise 2.10.4. To do...

Exercise 2.10.5. To do...

2.11. Hypergraph Regularity

Exercise 2.11.3. To do...

3. *Pseudorandom Graphs*

3.1. *Quasirandom Graphs*

Exercise 3.1.13. To do...

Exercise 3.1.17. To do...

Exercise 3.1.24. To do...

Exercise 3.1.27. To do...

Exercise 3.1.30. To do...

Exercise 3.1.31*. To do...

Exercise 3.1.32. To do...

Exercise 3.1.33*. To do...

Exercise 3.1.34*. To do...

3.2. *Expander Mixing Lemma*

Exercise 3.2.5. To do...

Exercise 3.2.8. To do...

Exercise 3.2.10. To do...

Exercise 3.2.14. To do...

Exercise 3.2.15. To do...

Exercise 3.2.16. To do...

Exercise 3.2.17. To do...

Exercise 3.2.18. To do...

3.3. *Abelian Cayley Graphs and Eigenvalues*

Exercise 3.3.16. To do...

Exercise 3.3.17. To do...

Exercise 3.3.18. To do...

Exercise 3.3.19*. To do...

3.4. *Quasirandom Groups*

Exercise 3.4.8. To do...

Exercise 3.4.11. To do. . .

3.5. Quasirandom Cayley Graphs and Grothendieck's Inequality

(This section has no exercises.)

3.6. Second Eigenvalue: Alon–Boppana Bound

Exercise 3.6.14. To do. . .

Exercise 3.6.15*. To do. . .

4. Graph Limits

4.1. Graphons

(This section has no exercises.)

4.2. Cut Distance

Exercise 4.2.11. To do...

4.3. Homomorphism Density

Exercise 4.3.10. To do...

4.4. W-Random Graphs

(This section has no exercises.)

4.5. Counting Lemma

(This section has no exercises.)

4.6. Weak Regularity Lemma

Exercise 4.6.12. To do...

Exercise 4.6.13*. To do...

Exercise 4.6.14. To do...

4.7. Martingale Convergence Theorem

(This section has no exercises.)

4.8. Compactness of the Graphon Space

Exercise 4.8.2. To do...

4.9. Equivalence of Convergence

Exercise 4.9.5. To do...

Exercise 4.9.7. To do...

Exercise 4.9.10. To do...

Exercise 4.9.11*. To do...

Exercise 4.9.12. To do...

5. *Graph Homomorphism Inequalities*

Exercise 5.0.8. To do...

Exercise 5.0.10. To do...

Exercise 5.0.11. To do...

Exercise 5.0.12. To do...

Exercise 5.0.13. To do...

5.1. *Edge vs. Triangle Densities*

Exercise 5.1.9. To do...

5.2. *Cauchy–Schwarz*

Exercise 5.2.4. To do...

Exercise 5.2.14. To do...

Exercise 5.2.15. To do...

Exercise 5.2.16. To do...

Exercise 5.2.17. To do...

Exercise 5.2.18. To do...

Exercise 5.2.19. To do...

5.3. *Hölder*

Exercise 5.3.8. To do...

Exercise 5.3.9. To do...

Exercise 5.3.12. To do...

Exercise 5.3.17. To do...

Exercise 5.3.21. To do...

Exercise 5.3.22. To do...

Exercise 5.3.23. To do...

Exercise 5.3.24. To do...

5.4. *Lagrangian*

Exercise 5.4.7. To do...

Exercise 5.4.8. To do...

Exercise 5.4.9. To do...

Exercise 5.4.10. To do...

Exercise 5.4.11. To do...

Exercise 5.4.12*. To do...

5.5. *Entropy*

Exercise 5.5.2. To do...

Exercise 5.5.13. To do...

Exercise 5.5.15. To do...

Exercise 5.5.18. To do...

Exercise 5.5.19. To do...

Exercise 5.5.21. To do...

Exercise 5.5.22. To do...

6. Forbidding 3-Term Arithmetic Progressions

6.1. Fourier Analysis in Finite Field Vector Spaces

Exercise 6.1.11. To do. . .

6.2. Roth's Theorem in the Finite Field Model

Exercise 6.2.11. To do. . .

Exercise 6.2.12. To do. . .

Exercise 6.2.13. To do. . .

Exercise 6.2.14. To do. . .

6.3. Fourier Analysis in the Integers

Exercise 6.3.7. To do. . .

Exercise 6.3.8. To do. . .

Exercise 6.3.9. To do. . .

6.4. Roth's Theorem in the Integers

Exercise 6.4.8*. To do. . .

6.5. Polynomial Method

Exercise 6.5.12. To do. . .

Exercise 6.5.13. To do. . .

6.6. Arithmetic Regularity

Exercise 6.6.2. To do. . .

Exercise 6.6.13. To do. . .

Exercise 6.6.15. To do. . .

6.7. Popular Common Difference

Exercise 6.7.2. To do. . .

Exercise 6.7.4. To do. . .

7. *Structure of Set Addition*

7.1. Sets of Small Doubling: Freiman's Theorem

Exercise 7.1.2. To do. . .

7.2. Sumset Calculus I: Ruzsa Triangle Inequality

Exercise 7.2.4. To do. . .

7.3. Sumset Calculus II: Plünnecke's Inequality

Exercise 7.3.7*. To do. . .

Exercise 7.3.8*. To do. . .

Exercise 7.3.9*. To do. . .

7.4. Covering Lemma

Exercise 7.4.3*. To do. . .

7.5. Freiman's Theorem in Groups with Bounded Exponent

Exercise 7.5.7. To do. . .

Exercise 7.5.8*. To do. . .

7.6. Freiman Homomorphisms

(This section has no exercises.)

7.7. Modeling Lemma

Exercise 7.7.4. To do. . .

Exercise 7.7.5. To do. . .

Exercise 7.7.6. To do. . .

7.8. Iterated Sumsets: Bogolyubov's Lemma

Exercise 7.8.7. To do. . .

Exercise 7.8.8. To do. . .

7.9. Geometry of Numbers

(This section has no exercises.)

7.10. *Finding a GAP in a Bohr Set*

(This section has no exercises.)

7.11. *Proof of Freiman's Theorem*

Exercise 7.11.2. To do. . .

7.12. *Polynomial Freiman–Ruzsa Conjecture*

Remark. The polynomial Freiman–Rusza conjecture has been proven by W. T. Gowers, Ben Green, Freddie Manners, and Terence Tao [To appear in *Annals of Mathematics*]. □

7.13. *Additive Energy and the Balog–Szemerédi–Gowers Theorem*

(This section has no exercises.)

8. Sum-Product Problem

8.1. Multiplication Table Problem

(This section has no exercises.)

8.2. Crossing Number Inequality and Point-Line Incidences

Exercise 8.2.8. To do. . .

8.3. Sum-Product via Multiplicative Energy

(This section has no exercises.)

9. *Progressions in Sparse Pseudorandom Sets*

9.1. *Green–Tao Theorem*

(This section has no exercises.)

9.2. *Relative Szemerédi Theorem*

(This section has no exercises.)

9.3. *Transference Principle*

(This section has no exercises.)

9.4. *Dense Model Theorem*

Exercise 9.4.1. To do. . .

Exercise 9.4.12. To do. . .

9.5. *Sparse Counting Lemma*

(This section has no exercises.)

9.6. *Proof of the Relative Roth Theorem*

Exercise 9.6.2. To do. . .