How many spanning trees in a bipartite graph?

Ho Boon Suan

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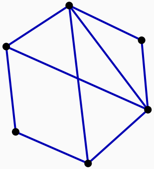
Graphs, trees, and spanning trees

- A graph is a set of vertices joined by edges.



A graph

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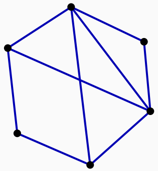
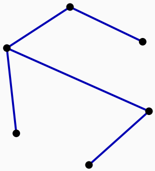


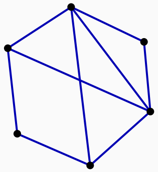
A graph



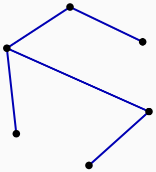
A tree

Graphs, trees, and spanning trees

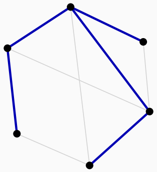
- A graph is a set of vertices joined by edges.
- A tree is a connected graph with no cycles.
- A spanning tree of a graph G is a tree that uses every vertex of G and only edges of G.



A graph

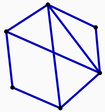


A tree

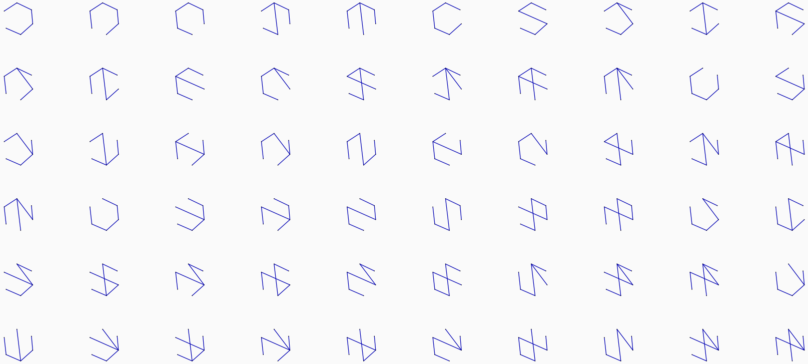


A spanning tree of the graph

Counting spanning trees



This graph has 60 spanning trees.



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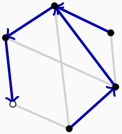
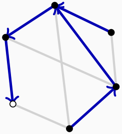


Ignore one vertex, then choose an edge out of each remaining vertex.

Counting spanning trees



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$$3 \times 4 \times 2 \times 4 \times 3 = 288 \text{ trees?}$$

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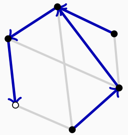

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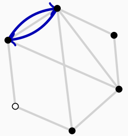

No, since some contain cycles.

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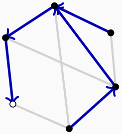

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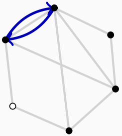
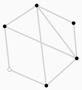


No, since some contain cycles.

$$-(2 \times 4 \times 3) = -24 \text{ trees}$$

Adding and subtracting graphs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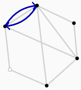
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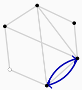
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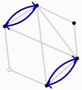
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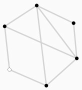
$3 \times 4 \times 2 \times 4 \times 3$ $\qquad$ $2 \times 4 \times 3$ $\qquad$ $3 \times 4 \times 2$ $\qquad$ 2

Adding and subtracting graphs



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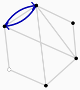

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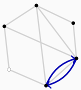

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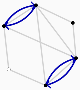
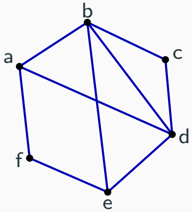


2

$- \cdots$

$$\text{spanning trees } \tau(G) = \sum_{\pi \in S_{n-1}} (-1)^{\#\text{cycles}} \prod_{\substack{v \text{ not in} \\ \text{a cycle}}} \deg(v)$$

The Laplacian matrix



$$L_G = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{bmatrix} 3 & -1 & 0 & -1 & 0 & -1 \\ -1 & 4 & -1 & -1 & -1 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ -1 & -1 & -1 & 4 & -1 & 0 \\ 0 & -1 & 0 & -1 & 3 & -1 \\ -1 & 0 & 0 & 0 & -1 & 2 \end{bmatrix} \end{array}$$

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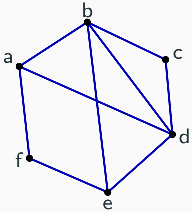

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Kirchhoff's matrix-tree theorem

Delete any row and its corresponding column from L_G.
The determinant of the resulting matrix is the number $\tau(G)$ of spanning trees of G.

The matrix-tree theorem

Corollary

Let G be a connected graph on n vertices, and let $0, \lambda_1, \ldots, \lambda_{n-1}$ be the eigenvalues of its Laplacian L_G. Then $\tau(G) = \frac{1}{n}\lambda_1\lambda_2 \ldots \lambda_{n-1}$.

The matrix-tree theorem

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Proof sketch.

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Proof sketch.

- Because the rows of L_G sum to zero, $\mathbf{1} = (1, \ldots, 1)^\top$ is an eigenvector with eigenvalue 0.

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- $[t] \det(tI - L_G) = (-1)^{n-1} \sum_{r=1}^{n} \det((L_G)_{\hat{r}}) = (-1)^{n-1} n\tau(G)$.

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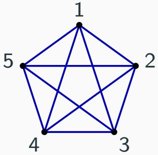

$$L_{K_5} = \begin{array}{c c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & -1 & 4 \end{bmatrix} \end{array}$$

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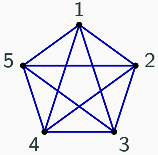

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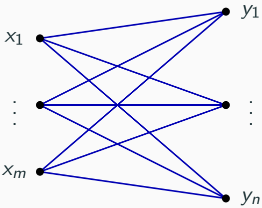
Eigenvalues of L_{K_5}: $0, 5, 5, 5, 5$, so $\tau(K_n) = \frac{1}{n}\, n^{n-1} = n^{n-2}$.

The matrix-tree theorem



$$L_{K_{m,n}} = \begin{pmatrix} nI_m & -J_{m\times n} \\ -J_{n\times m} & mI_n \end{pmatrix}$$

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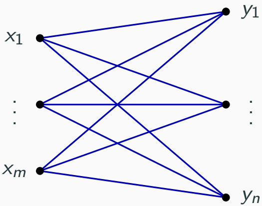

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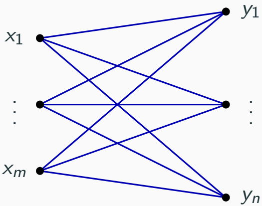

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How to calculate $\det(M)$?

Schur complements

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$$\det M = \det A \cdot \det S$$

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$$\begin{aligned} \det M &= \det(mI_{n-1}) \cdot \det\left(nI_m - \frac{1}{m} J_{m\times(n-1)} J_{(n-1)\times m} \right) \\ &= m^{n-1} \det\left(nI_m - \frac{n-1}{m} J_m \right) \end{aligned}$$

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Since eigenvalues of J_m are $m^{(1)}, 0^{(m-1)}$,

eigenvalues of $nI_m - \dfrac{n-1}{m} J_m$ are $1^{(1)}, n^{(m-1)}$.

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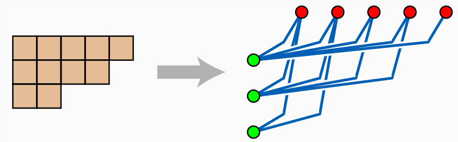
Thus $\tau(K_{m,n}) = m^{n-1} n^{m-1} = \frac{1}{mn} \prod_{v \in V(G)} \deg(v)$.

Demo

`https://web.archive.org/web/20260406060620/https://bipartite-spanning-trees-302294203391.us-west1.run.app/`

Ferrers graphs

A bipartite graph is a Ferrers graph if the neighborhoods of vertices in one part are linearly ordered by inclusion.



Theorem (Ehrenborg and van Willigenburg, 2004)

If $G = (X \sqcup Y, E)$ is a Ferrers graph, then

$$\tau(G) = \frac{1}{|X||Y|} \prod_{v \in V(G)} \deg(v).$$

Ferrers bound conjecture

Conjecture (Ehrenborg, 2006)

For every connected bipartite graph $G = (X \sqcup Y, E)$,

$$\tau(G) \leq \frac{1}{|X||Y|} \prod_{v \in V(G)} \deg(v).$$

Moreover, equality holds if and only if G is Ferrers.

Ferrers bound conjecture

Theorem (H., 2026)

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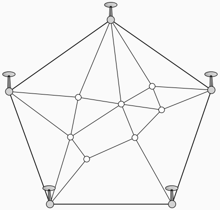
Rubber band representation



$$u_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix}$$

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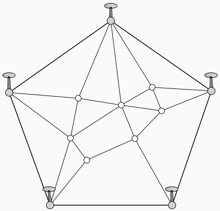

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$$\mathcal{E}(u) = \sum_{ij \in E} |u_i - u_j|^2 = \sum_{ij \in E} \big((x_i - x_j)^2 + (y_i - y_j)^2\big)$$

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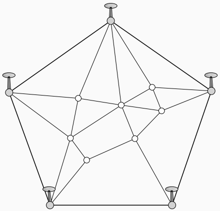

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minimize $\mathcal{E}(u)$
subject to $u_i = \bar{u}_i$ for all $i \in S$

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When energy is minimized, we have $\partial_i \mathcal{E}(u) = 0$:

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Thus $u_i = \frac{1}{\deg(i)} \sum_{j \in N(i)} u_j$.

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Proposition

$\mathcal{E}(u)$ is precisely the Laplacian quadratic form $u^\top L_G u$.

Proof. In general, $x^\top M x = \sum_{i,j} M_{ij} x_i x_j$; use the definition of L_G.

Energy minimization: The bipartite case

Notation

$$G = (X \sqcup Y, E)$$

$$X = \{x_1, \ldots, x_m\}, \; Y = \{y_1, \ldots, y_n\}$$

$$a_i = \deg(x_i), \quad b_j = \deg(y_j)$$

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$$\mathcal{E}(u, v) = \begin{pmatrix} u \\ v \end{pmatrix}^\top L_G \begin{pmatrix} u \\ v \end{pmatrix} = \sum_{x_i y_j \in E} (u_i - v_j)^2$$

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Fix u and find v minimizing $\mathcal{E}(u,v)$

Since $\mathcal{E}(u,v) = u^\top Au - 2u^\top Bv + v^\top Cv$, we may complete the square in v as follows: As

$$(v - Ku)^\top C(v - Ku) = v^\top Cv - 2u^\top K^\top Cv + u^\top K^\top CKu,$$

to match the cross term $-2u^\top Bv$, we choose $K = C^{-1}B^\top$ to get

$$\mathcal{E}(u,v) = u^\top(A - BC^{-1}B^\top)u + (v - Ku)^\top C(v - Ku).$$

Thus choosing $v = Ku$ minimizes $\mathcal{E}(u,v)$.

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$$\min_{v \in \mathbb{R}^n} \mathcal{E}(u,v) = u^\top L_X u = \sum_{x_i y_j \in E} (u_i - \bar{u}_{T_j})^2,$$

where $L_X = A - BC^{-1}B^\top$ is the Schur complement of block X. The v_i minimizing $\mathcal{E}(u,v)$ are the averages of neighborhood values: $v_i = \bar{u}_{T_i}$, where $T_i = N(y_i)$ and $\bar{u}_T := \frac{1}{|T|}\sum_{x_i \in T} u_i$.

Sum of projections

Use bipartite structure to sum over neighborhoods of Y-vertices:

$$\begin{aligned} u^\top L_X u &= \sum_{x_i y_j \in E} (u_i - \bar{u}_{T_j})^2 \\ &= \sum_{j=1}^{n} \sum_{x_i \in T_j} (u_i - \bar{u}_{T_j})^2 \\ &= \sum_{j=1}^{n} u^\top P_{T_j} u. \end{aligned}$$

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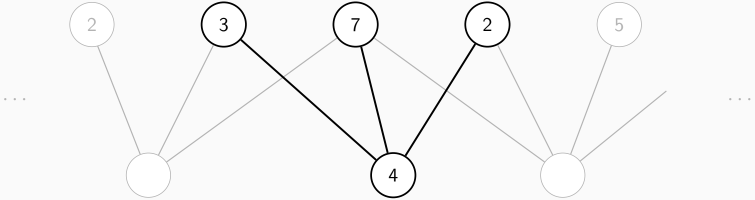
Given $T \subseteq X$, P_T is an orthogonal projection onto the subspace $H_T := \{u \in \mathbb{R}^m : \mathrm{supp}(u) \subseteq T \text{ and } \sum_{x_i \in T} u_i = 0\}$.

Sum of projections



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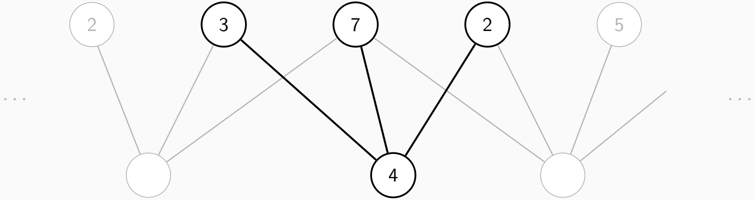

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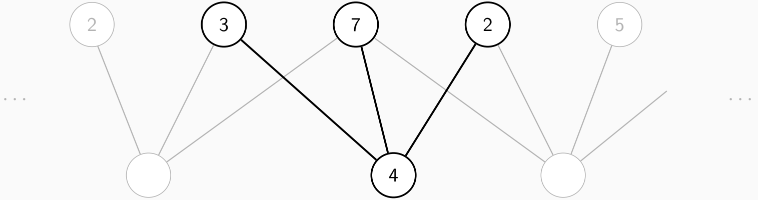

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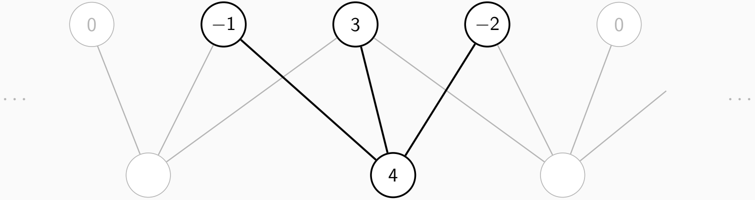

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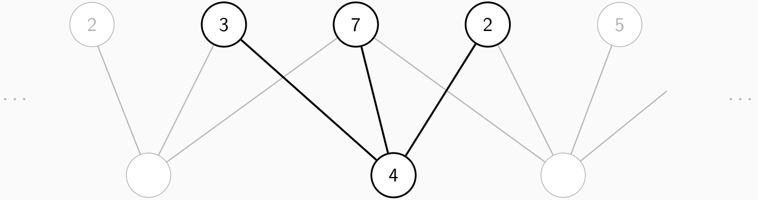

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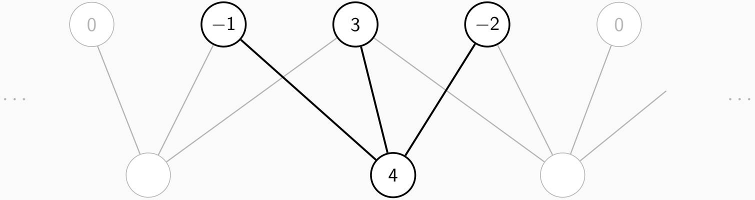

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- Hence u is constant on each neighborhood T_j. Connectedness then forces u to be constant on all of X.

Getting rid of the zero eigenvalue

Let $J = \mathbf{1}\mathbf{1}^\top$ be the $m \times m$ all-ones matrix. For every nonempty $T \subseteq X$, define

$$Q_T := P_T + \frac{1}{m} J.$$

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Therefore

$$M := L_X + \frac{n}{m} J = \sum_{j=1}^{n} \left(P_{T_j} + \frac{1}{m} J \right) = \sum_{j=1}^{n} Q_{T_j}.$$

Since $L_X \mathbf{1} = 0$, $J\mathbf{1} = m\mathbf{1}$, and $Ju = 0$ for $u \perp \mathbf{1}$, M is positive definite with eigenvalues $n, \mu_2, \ldots, \mu_m$.

Back to the matrix-tree theorem

Proposition

Let $M := L_X + \frac{n}{m}J$. Then $\tau(G) = \frac{1}{mn}\left(\prod_{j=1}^{n} b_j\right)\det M$.

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Since $\det(C) = \prod_{j=1}^n b_j$ and $\det M = n \prod_{i=2}^m \mu_i$, the proposition follows.

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Then concavity of $\log t$ implies $\sum_{i=1}^m \log \lambda_i \leq \sum_{i=1}^m \log a_i$, so $\det(M) = \prod_{i=1}^m \lambda_i \leq \prod_{i=1}^m a_i$.

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Karamata's inequality

If $(\lambda_1, \ldots, \lambda_m)$ majorizes $(a_1, \ldots, a_m)$, and if $\phi\colon (0, \infty) \to \mathbb{R}$ is strictly concave, then $\sum_{i=1}^{m} \phi(\lambda_i) \leq \sum_{i=1}^{m} \phi(a_i)$. Equality holds iff $\lambda_i = a_i$ for all i, assuming (λ) and (a) are weakly decreasing.

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- $(5, 1) \succeq (3, 3)$ since $5 > 3$ and $5 + 1 = 3 + 3$. Let $\phi(t) = \sqrt{t}$. Then $\sqrt{5} + \sqrt{1} \approx 3.236$, and $\sqrt{3} + \sqrt{3} \approx 3.464$.

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    - $\sqrt{5} + \sqrt{3} + \sqrt{1} \approx 4.968$
    - $\sqrt{4} + \sqrt{3} + \sqrt{2} \approx 5.146$
    - $\sqrt{3} + \sqrt{3} + \sqrt{3} \approx 5.196$

The endgame

Karamata's inequality

If $(\lambda_1, \ldots, \lambda_m)$ majorizes $(a_1, \ldots, a_m)$, and if $\phi\colon (0, \infty) \to \mathbb{R}$ is strictly concave, then $\sum_{i=1}^m \phi(\lambda_i) \le \sum_{i=1}^m \phi(a_i)$. Equality holds iff $\lambda_i = a_i$ for all i, assuming (λ) and (a) are weakly decreasing.

- $(5, 1) \succeq (3, 3)$ since $5 > 3$ and $5 + 1 = 3 + 3$. Let $\phi(t) = \sqrt{t}$. Then $\sqrt{5} + \sqrt{1} \approx 3.236$, and $\sqrt{3} + \sqrt{3} \approx 3.464$.
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- Majorized vector is more balanced; concavity rewards balance.

The endgame

Ky Fan's maximum principle

Let M be a real symmetric $m \times m$ matrix with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_m$. Then for each $1 \leq k \leq m$,

$$\sum_{i=1}^{k} \lambda_i = \max\big\{\mathrm{tr}(PM) : P \text{ is an orthogonal projection of rank } k\big\}.$$

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- If P projects onto $\mathrm{span}\{v_1, \ldots, v_k\}$ with (v_j) an orthonormal basis, then $\mathrm{tr}(PM) = \sum_{j=1}^{k} \langle M v_j, v_j \rangle$.

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- $M u_i = \lambda_i u_i$ for orthonormal eigenbasis (u_i), and $d_i = \langle P u_i, u_i \rangle = \|P u_i\|^2 \in [0, 1]$ satisfies $\sum_{i=1}^{m} d_i = \operatorname{tr}(P) = k$.

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- $Mu_i = \lambda_i u_i$ for orthonormal eigenbasis (u_i), and $d_i = \langle Pu_i, u_i \rangle = \|Pu_i\|^2 \in [0,1]$ satisfies $\sum_{i=1}^{m} d_i = \mathrm{tr}(P) = k$.
- Moreover, $\mathrm{tr}(PM) = \sum_{i=1}^{m} \lambda_i d_i$, so maximizing this gives $d_1 = \cdots = d_k = 1$ and $d_{k+1} = \cdots = d_m = 0$.

The endgame

Lemma

Recall that $Q_T = P_T + \frac{1}{m}J$. For nonempty subsets $I, T \subseteq X$,

$$\mathrm{tr}(Q_I Q_T) = |I \cap T| + \frac{|I \setminus T||T \setminus I|}{|I||T|} \geq |I \cap T|.$$

Equality holds if and only if $I \subseteq T$ or $T \subseteq I$.

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Proof. Compute, using the fact that $P_T = \operatorname{diag}(\mathbf{1}_T) - \frac{1}{|T|}\mathbf{1}_T\mathbf{1}_T^\top$.

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Remark

The overlap defect $\varepsilon(I, T) := \frac{|I \setminus T||T \setminus I|}{|I||T|}$ is the mechanism behind the equality case, since Ferrers graphs are precisely the connected bipartite graphs such that neighborhoods of vertices on one side are ordered by inclusion.

Majorization

Let $[k] \coloneqq \{x_1, \ldots, x_k\}$. Then, for $1 \leq k < m$,

$$\sum_{i=1}^{k} \lambda_i \geq \operatorname{tr}(Q_{[k]} M)$$

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Majorization

When $k = m$, we have

$$\sum_{i=1}^{m} \lambda_i = \mathrm{tr}(M) = \sum_{j=1}^{n} \mathrm{tr}(Q_{T_j}) = \sum_{j=1}^{n} |T_j| = \sum_{i=1}^{m} a_i.$$

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Thus $(\lambda_1, \ldots, \lambda_m)$ majorizes $(a_1, \ldots, a_m)$, so

$$\tau(G) \leq \frac{1}{mn} \prod_{v \in V(G)} \deg(v).$$

The equality case

Suppose equality holds. Earlier we saw for $1 \leq k < m$ that

$$\sum_{i=1}^{k} \lambda_i \geq \sum_{i=1}^{k} a_i + \sum_{j=1}^{n} \varepsilon([k], T_j) \geq \sum_{i=1}^{k} a_i.$$

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- If $t \geq 2$, then $x_t \in T_j$ means $T_j \not\subseteq [t-1]$, so $[t-1] \subseteq T_j$. Together with $x_t \in T_j$, this gives $T_j = [t]$.

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Since every T_j is an initial segment, the neighborhoods are ordered under inclusion, so G is Ferrers.

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$$M = \sum_{j=1}^{n} Q_{T_j} = \sum_{j=1}^{n} Q_{[t_j]}$$

is diagonal, with diagonal entries

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Thank you!

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boonsuan.github.io/ferrers-slides.pdf