

A SQUAREFREE LOWER BOUND FOR ERDŐS PROBLEM 675

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ABSTRACT. Let \mathcal{S} be the set of squarefree positive integers. For $N \geq 1$, call $t \geq 1$ admissible if

$$\mathbf{1}_{\mathcal{S}}(a+t) = \mathbf{1}_{\mathcal{S}}(a) \quad (1 \leq a \leq N).$$

We prove that, for every fixed positive $c < 25/72$ and all sufficiently large N , every admissible t is divisible by $\prod_{p \leq N^c} p^2$. Hence every such t , and in particular the least admissible translate, is larger than $\exp(N^c)$. Together with the standard translation property for the squarefree set, proved below for completeness, this answers the squarefree growth subquestion in Erdős Problem 675. The main input is Nunes's published refinement of Heath-Brown's bound for the least squarefree integer in a reduced residue class; the final exponential estimate uses the prime number theorem.

1. INTRODUCTION

Erdős Problem 675, as recorded by Bloom [1], asks about finite repetitions of membership patterns. A set $A \subseteq \mathbb{N}$ has the translation property if, for every $N \geq 1$, there is a positive integer t_N such that

$$a \in A \iff a + t_N \in A \quad (1 \leq a \leq N). \quad (1.1)$$

One part of the problem asks, when A is the set of squarefree integers, how quickly the least such t_N must grow, and in particular whether $t_N > \exp(N^c)$ for some absolute constant $c > 0$. The same problem page also records that elementary sieve theory implies the squarefree set has the translation property [1]. We include the standard proof of that fact in Proposition 1.1 below.

This note proves that the answer to this squarefree growth subquestion is yes. In fact, it proves a uniform divisibility statement for every translate that repeats the first N values of the squarefree indicator. The proof of this divisibility statement does not use the existence of admissible translates.

Let

$$T_{\mathcal{S}}(N) := \min\{t \geq 1 : \mathbf{1}_{\mathcal{S}}(a+t) = \mathbf{1}_{\mathcal{S}}(a) \text{ for all } 1 \leq a \leq N\}.$$

For the squarefree set this minimum is nonempty by Proposition 1.1.

Proposition 1.1. *The set \mathcal{S} of squarefree positive integers has the translation property.*

Proof. Fix $N \geq 1$, and let

$$B := \{1 \leq a \leq N : a \in \mathcal{S}\}, \quad k := |B|, \quad P := \prod_{p \leq N} p^2.$$

It is enough to find an integer $m \geq 1$ such that $Pm + a$ is squarefree for every $a \in B$. Indeed, then $t = Pm$ works: if $a \leq N$ is not squarefree, then $p^2 \mid a$ for some prime $p \leq N$, whence $p^2 \mid a + t$; while if $a \in B$, the chosen property of m gives $a + t \in \mathcal{S}$.

For later use, note that if $a \in B$ and $p \leq N$ is prime, then

$$Pm + a \equiv a \not\equiv 0 \pmod{p^2},$$

because a is squarefree. Thus, when $a \in B$, the only possible square obstructions to $Pm + a$ come from primes $q > N$.

For primes $q > N$, the congruences

$$q^2 \mid Pm + a \quad (a \in B)$$

exclude exactly k residue classes modulo q^2 : here $(P, q) = 1$, and the classes are distinct because the elements of B lie between 1 and $N < q^2$. For $z > N$, let $S_z(X)$ be the number of integers $1 \leq m \leq X$ for which none of these exclusions occurs for any prime $N < q \leq z$. By the Chinese remainder theorem,

$$S_z(X) = X\delta_z + O_{N,z}(1), \quad \delta_z := \prod_{N < q \leq z} \left(1 - \frac{k}{q^2}\right),$$

where the product is over primes. Since $\sum_q k/q^2 < \infty$, the products δ_z tend to a positive limit $\delta = \prod_{q > N} (1 - k/q^2) > 0$.

Let $S(X)$ count the integers $1 \leq m \leq X$ for which $Pm + a$ is squarefree for every $a \in B$. The only obstruction not counted by $S_z(X)$ is the occurrence of some prime $q > z$ with $q^2 \mid Pm + a$ for at least one $a \in B$. For fixed $q > N$ and $a \in B$, there is one residue class modulo q^2 for m , so there are at most $X/q^2 + 1$ such $m \leq X$. Also such a prime must satisfy $q \leq (PX + N)^{1/2}$. Hence

$$S(X) \geq S_z(X) - k \sum_{z < q \leq (PX+N)^{1/2}} \left(\frac{X}{q^2} + 1\right).$$

Dividing by X ,

$$\frac{S(X)}{X} \geq \frac{S_z(X)}{X} - k \sum_{z < q \leq (PX+N)^{1/2}} \frac{1}{q^2} - k \frac{\pi((PX+N)^{1/2})}{X},$$

where the sums over q are over primes. Letting $X \rightarrow \infty$, with N, P, z fixed, the last term tends to zero and $S_z(X)/X \rightarrow \delta_z$. Therefore

$$\liminf_{X \rightarrow \infty} \frac{S(X)}{X} \geq \delta_z - k \sum_{q > z} \frac{1}{q^2}.$$

Choose z so large that the right-hand side is positive. Then $S(X) > 0$ for all sufficiently large X , and therefore such an m exists. This completes the proof. \square

Theorem 1.2. *Let $0 < c < 25/72$. There is an integer $N_0(c)$ such that, whenever $N \geq N_0(c)$ and $t \geq 1$ satisfies*

$$\mathbf{1}_S(a+t) = \mathbf{1}_S(a) \quad (1 \leq a \leq N), \tag{1.2}$$

one has

$$\prod_{p \leq N^c} p^2 \mid t, \tag{1.3}$$

where the product is over primes. Consequently $t > \exp(N^c)$. In particular,

$$T_S(N) > \exp(N^c) \quad (N \geq N_0(c)).$$

Taking, for example, $c = 1/3$ gives the lower bound asked for in the squarefree part of the problem. The exponent $25/72$ is not claimed to be optimal; it is the exponent obtained from the published Nunes least-squarefree-in-progressions input quoted in Section 2.

The proof is local at each small prime. If $p \leq N^c$ and $p^2 \nmid t$, we construct a squarefree integer $a \leq N$ such that $p^2 \mid a + t$. Then a is squarefree while $a + t$ is not, contradicting (1.2). The construction requires finding small squarefree integers in prescribed reduced residue classes modulo p and p^2 .

2. THE SQUAREFREE INPUT

For a modulus $q \geq 1$ and a reduced residue class $r \pmod{q}$, write $L(q, r)$ for the least positive squarefree integer m such that $m \equiv r \pmod{q}$.

Theorem 2.1 (Nunes, published Corollary 1.2). *For every $\varepsilon > 0$ there is a constant C_ε such that, uniformly for all $q \geq 1$ and all r with $(r, q) = 1$,*

$$L(q, r) \leq C_\varepsilon q^{36/25+\varepsilon}.$$

This is Corollary 1.2 of the published version of Nunes [3, Corollary 1.2, p. 484], in the notation $n(q, a)$ for the least positive squarefree integer congruent to $a \pmod{q}$. Heath-Brown's earlier bound $n(q, a) \ll q^{13/9+\varepsilon}$ is recalled immediately before that corollary in [3, p. 484] and is originally due to [2].

We emphasize that the input used here is the published *Mathematika* version of Nunes, specifically Corollary 1.2; the arXiv preprint is worded differently around the squarefree-modulus issue and should not be used for this particular citation. The published Corollary 1.2 is stated uniformly for all coprime pairs $(a, q) = 1$, with no squarefree-modulus hypothesis on q . The published paper also makes the relevant non-squarefree input explicit in Lemma 1.4, which treats moduli q that are either squarefree or the square of a prime [3, Lemma 1.4, p. 485]. Thus Theorem 2.1 applies to the moduli $q = p$ and $q = p^2$ used below.

The crucial case is $q = p^2$. There Theorem 2.1 gives

$$L(p^2, r) \ll_\varepsilon (p^2)^{36/25+\varepsilon} = p^{72/25+2\varepsilon},$$

and this is what produces the threshold $25/72$.

3. PROOF OF THE THEOREM

The key point is the following uniform local divisibility statement.

Lemma 3.1. *Fix $0 < c < 25/72$. There is an integer $N_1(c)$ such that, whenever $N \geq N_1(c)$, $t \geq 1$ satisfies (1.2), and $p \leq N^c$ is prime, then $p^2 \mid t$.*

Proof. Choose $\varepsilon > 0$ so small that

$$\alpha_1 := c \left(\frac{72}{25} + 2\varepsilon \right) < 1. \tag{3.1}$$

Then also

$$\alpha_2 := c \left(\frac{61}{25} + \varepsilon \right) < 1, \quad (3.2)$$

since $61/25 + \varepsilon < 72/25 + 2\varepsilon$. Let C_ε be the constant in Theorem 2.1. Choose $N_1(c)$ so large that, for all $N \geq N_1(c)$,

$$C_\varepsilon N^{\alpha_1} \leq N \quad \text{and} \quad C_\varepsilon N^{\alpha_2} \leq N. \quad (3.3)$$

This lower bound for N depends only on c , through the chosen ε and the corresponding constant C_ε .

Let $p \leq N^c$ be prime, and suppose for contradiction that $p^2 \nmid t$. There are two cases.

Case 1: $p \nmid t$. The residue class $-t \pmod{p^2}$ is reduced. By Theorem 2.1, there is a positive squarefree integer a such that

$$a \equiv -t \pmod{p^2}, \quad a \leq C_\varepsilon (p^2)^{36/25+\varepsilon} = C_\varepsilon p^{72/25+2\varepsilon}.$$

Since $p \leq N^c$, equations (3.1) and (3.3) give

$$a \leq C_\varepsilon N^{\alpha_1} \leq N.$$

Thus $1 \leq a \leq N$ and a is squarefree. But $a + t \equiv 0 \pmod{p^2}$, so $a + t$ is not squarefree. This contradicts (1.2).

Case 2: $p \mid t$, but $p^2 \nmid t$. Write $t = pu$, with $p \nmid u$. The residue class $-u \pmod{p}$ is reduced. By Theorem 2.1, there is a positive squarefree integer b such that

$$b \equiv -u \pmod{p}, \quad b \leq C_\varepsilon p^{36/25+\varepsilon}.$$

Because $b \equiv -u \not\equiv 0 \pmod{p}$, we have $p \nmid b$. Hence $a := pb$ is squarefree. Moreover, using $p \leq N^c$ and (3.3),

$$a = pb \leq C_\varepsilon p^{61/25+\varepsilon} \leq C_\varepsilon N^{\alpha_2} \leq N.$$

Finally,

$$a + t = pb + pu = p(b + u),$$

and $b + u \equiv 0 \pmod{p}$, so $p^2 \mid a + t$. Again a is squarefree while $a + t$ is not, contradicting (1.2).

Both alternatives lead to contradictions. Therefore $p^2 \mid t$, as claimed. \square

Proof of Theorem 1.2. Let $N_2(c)$ be large enough that

$$2\vartheta(N^c) > N^c \quad (N \geq N_2(c)), \quad (3.4)$$

where $\vartheta(x) := \sum_{p \leq x} \log p$. Such an $N_2(c)$ exists by the prime number theorem, $\vartheta(x) \sim x$. Put $N_0(c) := \max\{N_1(c), N_2(c)\}$.

Now let $N \geq N_0(c)$, and let t satisfy (1.2). By Lemma 3.1, $p^2 \mid t$ for every prime $p \leq N^c$. These prime powers are pairwise coprime, and therefore

$$\prod_{p \leq N^c} p^2 \mid t.$$

This proves (1.3). Since t is positive,

$$\log t \geq 2 \sum_{p \leq N^c} \log p = 2\vartheta(N^c) > N^c,$$

where the last inequality is (3.4). Hence $t > \exp(N^c)$. \square

Corollary 3.2. *The squarefree growth subquestion in Erdős Problem 675 has a positive answer: for every fixed $0 < c < 25/72$, and in particular for $c = 1/3$, the least translate repeating the squarefree pattern on $[1, N]$ satisfies*

$$T_S(N) > \exp(N^c)$$

for all sufficiently large N .

Proof. Proposition 1.1 shows that $T_S(N)$ is defined, and Theorem 1.2 gives the stated lower bound for every admissible translate, hence for the least one. \square

4. REMARKS

Remark 4.1 (Scope). The argument proves only the squarefree lower bound above. It does not address the other parts of Erdős Problem 675, such as the translation property for sums of two squares or for integers supported on one part of a partition of the primes.

Remark 4.2 (Dependence on the least-squarefree exponent). The proof uses Theorem 2.1 as a black box. If, for the moduli needed here, one had instead a bound

$$L(q, r) \ll_{\varepsilon} q^{\kappa+\varepsilon}$$

for all reduced residue classes, then the same argument would force $p^2 \mid t$ for every prime $p \leq N^c$ whenever

$$c < \min \left\{ \frac{1}{2\kappa}, \frac{1}{1+\kappa} \right\}.$$

For the published Nunes exponent $\kappa = 36/25$, these two quantities are $25/72$ and $25/61$, so the first is the limiting one. With only Heath-Brown's earlier exponent $\kappa = 13/9$, the same argument would give $9/26$.

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- [3] R. M. Nunes, *On the least squarefree number in an arithmetic progression*, Mathematika **63** (2017), no. 2, 483–498; see especially Corollary 1.2 of the published version. The arXiv preprint <https://arxiv.org/abs/1605.03347> differs in wording around the squarefree-modulus issue.