NON-INTERSECTING ARITHMETIC PROGRESSIONS VIA SPREAD CORES

BOON SUAN HO

Abstract. Let $f(x)$ be the largest size of a set of moduli $q \le x$ for which one can choose pairwise disjoint residue classes $a_q \pmod q$. De la Bretèche, Ford and Vandehey proved

$$xL(-1+o(1),x) \le f(x) \le xL\big(-\sqrt{3}/2+o(1),x\big)\,, \qquad L(\alpha,x) := \exp\{\alpha\sqrt{\log x \log\log x}\},$$

and conjectured that the lower bound is sharp. We prove the conjecture. The analytic and pruning input is exactly that of de la Bretèche–Ford–Vandehey. The new ingredient is a spread-core lemma: every intersecting k-uniform family with $k \le K$ has a non-empty core C contained in at least a $(C_0 \log(eK))^{-|C|}$ fraction of the family. This logarithmic loss replaces the Erdős–Lovász minimal-family loss in the descending chain and yields

$$f(x) = xL(-1+o(1),x)\,.$$

1. Introduction

For $x \ge 1$, let $f(x)$ be the maximum cardinality of a set $\mathcal{Q} \subseteq [1,x] \cap \mathbb{N}$ for which there are residue classes

$$a_q \pmod q \qquad (q \in \mathcal{Q})$$

that are pairwise disjoint. Set

$$L(\alpha,x) := \exp\{\alpha\sqrt{\log x \log\log x}\}.$$

This is Erdős Problem #202 in Bloom's list [1]. Erdős and Stein conjectured that $f(x) = o(x)$; this was proved by Erdős and Szemerédi [6]. Subsequent quantitative work of Croot [5], Chen [4], and de la Bretèche, Ford and Vandehey [3] led to the best previously published estimate

$$xL(-1+o(1),x) \le f(x) \le xL\Big(-\sqrt{3}/2+o(1),x\Big)\,. \tag{1}$$

De la Bretèche–Ford–Vandehey conjectured that the lower bound gives the correct order. We prove this.

Theorem 1.1. *As $x \to \infty$,*

$$f(x) = xL(-1+o(1),x) = x\exp\{-(1+o(1))\sqrt{\log x \log\log x}\}.$$

As an immediate corollary, we also resolve Erdős Problem #1190 in Bloom's list [2] up to the same L-scale.

Corollary 1.2. *Let*

$$\epsilon_m := \sup\sum_{i=1}^{k}\frac{1}{n_i},$$

where the supremum is taken over all finite sequences $m < n_1 < \cdots < n_k$ for which there exist pairwise disjoint residue classes $a_i \pmod{n_i}$. Then, as $m \to \infty$,

$$\epsilon_m = L(-1+o(1),m) = \exp\{-(1+o(1))\sqrt{\log m \log\log m}\}.$$

The proof changes only one point in the upper-bound argument of [3]. In that argument one repeatedly passes to a subfamily whose moduli share an exact prime-power block and whose residues agree modulo that block. The remaining prime supports form an intersecting set system. De la Bretèche–Ford–Vandehey used an Erdős–Lovász bound for minimal intersecting families to find a common block, losing roughly K^w when the block contains w primes and the original moduli have K prime factors. We replace this by a Kahn–Kalai spread consequence, losing only $(C \log K)^w$. Since the BFV reduction has $K \ll \sqrt{\log x / \log\log x}$, the accumulated logarithmic loss is $e^{o(\log x)}$, and the final optimization gives the constant 1.

The paper is organized as follows. Section 2 proves the spread-core lemma. Section 3 records the BFV input in the form used below, including the needed uniformity of the $o(1)$ terms. Section 4 packages the modified descending chain into a single proposition. Section 5 performs the final optimization and proves Corollary 1.2.

2. The spread-core input

For a finite set family $\mathcal{F}$ and a set T, write

$$\mathcal{F}_T := \{F \in \mathcal{F} : T \subseteq F\}.$$

A k-uniform family $\mathcal{F}$ is κ-spread if

$$|\mathcal{F}_T| \le \frac{|\mathcal{F}|}{\kappa^{|T|}} \qquad (T \neq \varnothing). \tag{2}$$

We use the following standard consequence of the Park–Pham theorem [7].

Proposition 2.1 (Spread-disjointness). *There is an absolute constant C_{sp} such that, for every $r \ge 2$ and $k \ge 1$, every non-empty k-uniform κ-spread family with*

$$\kappa \ge C_{\mathrm{sp}} r \log(ek)$$

contains r pairwise disjoint members.

Proof. We give the short derivation from the expectation-threshold form of the Park–Pham theorem. Let X be the ground set and let $\langle \mathcal{F} \rangle$ be the increasing family of all subsets of X that contain a member of $\mathcal{F}$. An increasing family $\mathcal{U}$ is p-small if it is contained in $\langle \mathcal{G} \rangle$ for some $\mathcal{G}$ with $\sum_{G \in \mathcal{G}} p^{|G|} \le 1/2$. Let $q(\mathcal{U})$ be the supremum of such p. This is the same expectation-threshold parameter $q(\mathcal{U})$ appearing in Park and Pham's formulation. Let $p_c(\mathcal{U})$ be the 1/2 threshold under product measure. The Kahn–Kalai conjecture, proved by Park and Pham [7, Theorem 1.1], gives

$$p_c(\mathcal{U}) \le C_{\mathrm{KK}} q(\mathcal{U}) \log \ell(\mathcal{U}), \tag{3}$$

where $\ell(\mathcal{U})$ is the maximum of 2 and the largest size of an inclusion-minimal member of $\mathcal{U}$.

We claim that $q(\langle \mathcal{F} \rangle) \le \kappa^{-1}$. Indeed, suppose $\mathcal{G}$ covers $\langle \mathcal{F} \rangle$. Then each $F \in \mathcal{F}$ contains at least one $G \in \mathcal{G}$. Discard all G contained in no member of $\mathcal{F}$; if $\varnothing \in \mathcal{G}$, then the cost at $p = \kappa^{-1}$ is already at least 1. Otherwise (2) gives

$$|\mathcal{F}| \le \sum_{G \in \mathcal{G}} |\mathcal{F}_G| \le |\mathcal{F}| \sum_{G \in \mathcal{G}} \kappa^{-|G|}.$$

Thus no cover has cost at most 1/2 at $p = \kappa^{-1}$, so $q(\langle \mathcal{F} \rangle) \le \kappa^{-1}$. Since the minimal members of $\langle \mathcal{F} \rangle$ have size at most k, (3) gives

$$p_c(\langle \mathcal{F} \rangle) \le C_{\mathrm{KK}} \kappa^{-1} \log(ek).$$

With C_{sp} large enough, this is less than $1/(2r)$.

Randomly partition X into $2r$ parts. Each part has the product distribution with density $1/(2r)$, hence contains a member of $\mathcal{F}$ with probability at least $1/2$. The expected number of successful parts is therefore at least r. Some partition has at least r successful parts, and choosing one member of $\mathcal{F}$ from each successful part gives r pairwise disjoint members. □

Corollary 2.2 (Dense core). *There is an absolute constant C_0 with the following property. Let $\mathcal{A}$ be a non-empty intersecting family of distinct k-element sets, with $1 \le k \le K$. Then some non-empty set C satisfies*

$$|\mathcal{A}_C| > \frac{|\mathcal{A}|}{(C_0 \log(eK))^{|C|}}. \tag{4}$$

Proof. Choose $C_0 \ge 2C_{\mathrm{sp}}$. If (4) failed for every non-empty C, then $\mathcal{A}$ would be κ-spread with $\kappa = C_0 \log(eK) \ge C_{\mathrm{sp}} \cdot 2\log(ek)$. Proposition 2.1, with $r = 2$, would give two disjoint members of $\mathcal{A}$, contradicting that $\mathcal{A}$ is intersecting. □

Remark 2.3. The core loss in Corollary 2.2 is K-dependent and therefore weaker than the K-independent combinatorial conjecture formulated in [3, Conjecture 2]. It is nevertheless sufficient here: in the BFV reduction $K \ll \sqrt{\log x/\log\log x}$, and the total contribution of the factors $(C\log K)^w$ in the chain is $e^{o(\log x)}$.

3. Input from de la Bretèche–Ford–Vandehey

Throughout the proof set

$$X := \log x, \qquad Y := \log\log x, \qquad M := \sqrt{X/Y}, \qquad Z := \sqrt{XY}.$$

Thus $L(\alpha, x) = e^{\alpha Z}$, $Z/M = Y$, and $MZ = X$. For an integer n, write

$$\omega(n) := \sum_{p|n} 1, \qquad \mathrm{rad}(n) := \prod_{p|n} p, \qquad h(n) := \prod_{p^\nu \| n} \nu.$$

All $o(1)$ terms are as $x \to \infty$. Whenever they depend on auxiliary parameters below, the dependence is uniform for

$$0 \le d \le 3, \qquad 0 \le W \le K \le 3M, \qquad 1 \le r \le R \le K.$$

In particular, when a factor $L(\eta(x), x)$ with a uniform $\eta(x) = o(1)$ is used $O(M)$ times, its accumulated logarithm is $O(M)\eta(x)Z = o(X)$; this is the only uniformity needed in the final product estimate.

The following proposition is the pruning step of [3, Section 4.1, conditions (1)–(5)], stated in the form used here.

Proposition 3.1 (BFV pruning). *Let $\mathcal{Q} \subseteq [1, x]$ be an extremal family of moduli with pairwise disjoint residue classes, and write $S = |\mathcal{Q}| = f(x)$. Then there is a subfamily $\mathcal{Q}' \subseteq \mathcal{Q}$, of size S', such that*

- (P1) $S' \ge SL(o(1), x)$;
- (P2) $xL(-2, x) \le q \le x$ *for every* $q \in \mathcal{Q}'$;
- (P3) $h(q) \le e^{\sqrt{X}}$ *for every* $q \in \mathcal{Q}'$;
- (P4) $\omega(q) = K$ *for every* $q \in \mathcal{Q}'$, *for some* $K \le 3M$;
- (P5) *the squarefree kernels* $\mathrm{rad}(q)$, $q \in \mathcal{Q}'$, *are distinct.*

Proof. This is the cleanup carried out in [3, Section 4.1]. The known lower bound in (1) allows one to discard moduli below $xL(-2, x)$ and lose only a factor $L(o(1), x)$. Lemma 3.1 of [3], with $\alpha = 3$, discards moduli with more than $3M$ prime factors at the same cost. Lemma 3.2 of [3] gives

$$\#\{n \le x : h(n) > e^{\sqrt{X}}\} \ll x \exp\{-\tfrac{1}{5}\sqrt{X}\,Y\} = o\big(xL(-1,x)\big),$$

so condition (P3) can also be imposed without changing the main scale. Pigeonholing the value of $\omega(q)$ costs at most $O(M) = L(o(1),x)$. Finally, [3, Lemma 3.3] bounds the multiplicity of each squarefree kernel under (P3) and (P4) by

$$e^{2\sqrt{X}}2^K = L(o(1),x)\,,$$

so keeping one representative for each kernel gives the asserted subfamily. These are precisely the displayed conditions (1)–(5) in [3, Section 4.1], with rad in place of their notation ker. ☐

We also need one counting estimate, again from [3, Lemma 3.1]. In the form most convenient for the argument below, uniformly for every fixed $A > 0$, for $0 \le \alpha \le A$, and for $2 \le y \le x$,

$$\#\{n \le y : \omega(n) > \alpha M\} \le yL(-\alpha/2 + o(1), x)\,. \tag{5}$$

If one prefers the variant with $\ge$ rather than $>$ in the threshold, the two formulations differ only by shifting the threshold by one, which changes the right-hand side by at most a factor $(\log x)^{1/2} = L(o(1),x)$ in the ranges below. The cases $1 \le y < 2$ are trivial and will be absorbed into the same notation. We shall use the following quotient count.

Lemma 3.2. *Let $K = dM$ with $0 \le d \le 3$, and let $0 \le W \le K$. Uniformly for $1 \le y \le x$,*

$$\#\{n \le y : \omega(n) = K - W\} \le yL(-d/2 + o(1), x)\,(\log x)^{W/2}. \tag{6}$$

The $o(1)$ term is uniform for $0 \le d \le 3$, $0 \le W \le K$, and $1 \le y \le x$.

Proof. If $W = K$, then $\omega(n) = 0$ forces $n = 1$, so (6) is trivial. Assume therefore that $W < K$, and put $t := K - W \ge 1$. Then

$$\#\{n \le y : \omega(n) = t\} \le \#\{n \le y : \omega(n) > t - 1\}.$$

If $y < 2$, the required bound is again trivial. Otherwise apply (5) with

$$\alpha = \frac{t-1}{M} = d - \frac{W}{M} - \frac{1}{M}.$$

Since $Z/M = Y = \log\log x$,

$$\begin{aligned} L(-\alpha/2 + o(1), x) &= L(-d/2 + o(1), x) \exp\Big\{\frac{W+1}{2M} Z\Big\} \\ &= L(-d/2 + o(1), x)\,(\log x)^{W/2}(\log x)^{1/2}. \end{aligned}$$

The extra factor $(\log x)^{1/2} = L(o(1),x)$ is absorbed into the uniform $o(1)$ term, giving (6). ☐

We shall repeatedly use the elementary criterion that two residue classes $a \pmod q$ and $b \pmod r$ intersect if and only if

$$a \equiv b \pmod{(q,r)}. \tag{7}$$

Thus two chosen residue classes that are disjoint must fail this congruence for their moduli.

4. The chain inequality

Let $\mathcal{Q}'$ be a family satisfying Proposition 3.1. For each $q \in \mathcal{Q}'$, keep the chosen residue class $a_q \pmod q$. Put $S' = |\mathcal{Q}'|$, write $K = dM$ with $0 \le d \le 3$, and set

$$\Lambda := C_\Lambda \log(eK), \qquad C_\Lambda := \frac{\pi^2}{6} C_0,$$

where C_0 is the constant in Corollary 2.2. For large x the condition $q \ge xL(-2, x) > 1$ ensures $K \ge 1$.

Lemma 4.1 (Weighted pigeonholing). *Let I be a finite set, and let $(N_i)_{i\in I}$ and $(w_i)_{i\in I}$ be families of real numbers with $N_i \ge 0$ and $w_i > 0$. Write*

$$N := \sum_{i\in I} N_i, \qquad W := \sum_{i\in I} w_i.$$

Then some $i \in I$ satisfies

$$N_i \ge N \frac{w_i}{W}.$$

Proof. Otherwise $N_i < Nw_i/W$ for every i, and summing over I gives $N < N$. □

Proposition 4.2 (Chain inequality). *There is an integer R with $1 \le R \le K$ and pairwise coprime integers $P_1, \dots, P_R > 1$ such that, with*

$$P_{\le r} := P_1 \cdots P_r, \qquad W_r := \omega(P_{\le r}) \quad (1 \le r \le R), \qquad W_0 := 0,$$

one has $W_R = K$,

$$P_{\le R} \ge xL(-2, x)\,, \tag{8}$$

and, for every $1 \le r \le R$,

$$P_r \le \frac{x}{S'} L(-d/2 + o(1), x)\,(\log x)^{W_r/2} \Lambda^{W_r} e^{2\sqrt{X}}. \tag{9}$$

The $o(1)$ in (9) is uniform for $0 \le d \le 3$, $1 \le r \le R$, and $0 \le W_r \le K$.

Proof. We construct the P_r by induction. At stage r we have selected an exact product $P_{\le r-1}$ and a subfamily $\mathcal{Q}'_{r-1} \subseteq \mathcal{Q}'$ such that every $q \in \mathcal{Q}'_{r-1}$ is divisible by $P_{\le r-1}$, no prime of $P_{\le r-1}$ divides $q/P_{\le r-1}$, and all residues a_q agree modulo $P_{\le r-1}$. Initially $P_{\le 0} = 1$ and $\mathcal{Q}'_0 = \mathcal{Q}'$.

Assume $W_{r-1} < K$. For $q \in \mathcal{Q}'_{r-1}$ let

$$A(q) := \{p : p \mid q/P_{\le r-1}\}.$$

The sets $A(q)$ are distinct: if $A(q_1) = A(q_2)$, then

$$\mathrm{rad}(q_i) = \mathrm{rad}(P_{\le r-1}) \prod_{p\in A(q_1)} p \qquad (i = 1, 2),$$

so $\mathrm{rad}(q_1) = \mathrm{rad}(q_2)$, contrary to Proposition 3.1. They all have common size

$$|A(q)| = K - W_{r-1},$$

which lies in $[1, K]$ because $W_{r-1} < K$. They are also intersecting. Indeed, if $A(q_1)$ and $A(q_2)$ were disjoint, then (q_1, q_2) would divide the exact product $P_{\le r-1}$. Since the residues agree modulo $P_{\le r-1}$, they would agree modulo (q_1, q_2), and (7) would force the two residue classes to intersect, a contradiction.

Corollary 2.2 therefore gives a non-empty set of primes C_r, say $|C_r| = w_r$, contained in more than

$$\frac{|\mathcal{Q}'_{r-1}|}{(C_0 \log(eK))^{w_r}} \tag{10}$$

of the supports $A(q)$. Let

$$\mathcal{S}_r := \{q \in \mathcal{Q}'_{r-1} : C_r \subseteq A(q)\},$$

so (10) gives the same lower bound for $|\mathcal{S}_r|$.

For $q \in \mathcal{S}_r$ put

$$B_{C_r}(q) := \prod_{p \in C_r} p^{\nu_p(q)}.$$

For each attained value u of $B_{C_r}(q)$ on $\mathcal{S}_r$, let

$$N_u := \#\{q \in \mathcal{S}_r : B_{C_r}(q) = u\}.$$

Apply Lemma 4.1 with weights $w_u := h(u)^{-2}$. Since the attained values form a subset of the products

$$u = \prod_{p \in C_r} p^{\nu_p} \qquad (\nu_p \geq 1),$$

their total weight is at most

$$\prod_{p \in C_r} \sum_{\nu \geq 1} \frac{1}{\nu^2} = \left(\frac{\pi^2}{6}\right)^{w_r}.$$

Hence some attained value P_r satisfies

$$N_{P_r} \geq |\mathcal{S}_r| \frac{h(P_r)^{-2}}{(\pi^2/6)^{w_r}} > \frac{|\mathcal{Q}'_{r-1}|}{\Lambda^{w_r} h(P_r)^2}.$$

Let

$$\mathcal{Q}_r := \{q \in \mathcal{S}_r : B_{C_r}(q) = P_r\}.$$

Among the residues $a_q \pmod{P_r}$ with $q \in \mathcal{Q}_r$, choose the most frequent one and let $\mathcal{Q}'_r$ be the corresponding subfamily. Then

$$|\mathcal{Q}_r| > \frac{|\mathcal{Q}'_{r-1}|}{\Lambda^{w_r} h(P_r)^2}, \qquad |\mathcal{Q}'_r| \geq \frac{|\mathcal{Q}_r|}{P_r}. \tag{11}$$

Because the surviving moduli in $\mathcal{Q}_r$ satisfy $B_{C_r}(q) = P_r$, every prime in C_r is removed completely from $q/P_{\leq r-1}$; equivalently, no prime of P_r divides the remaining quotient $q/(P_{\leq r-1}P_r)$. Since the primes of P_r lie in the remaining support, P_r is coprime to $P_{\leq r-1}$; and because the surviving residues already agree modulo $P_{\leq r-1}$ and now also modulo P_r, they agree modulo $P_{\leq r} = P_{\leq r-1}P_r$. This preserves the induction hypotheses.

The process stops when $W_R = K$, after at most K steps. Then each $q \in \mathcal{Q}'_R$ is divisible by the exact product $P_{\leq R}$, no prime of $P_{\leq R}$ divides $q/P_{\leq R}$, and

$$\omega(q) = K = W_R = \omega(P_{\leq R}).$$

Hence $q/P_{\leq R}$ has no prime divisor, so $q = P_{\leq R}$. Thus $\mathcal{Q}'_R = \{P_{\leq R}\} \subseteq \mathcal{Q}'$, and Proposition 3.1 gives (8).

It remains to prove (9). Write

$$w_j := W_j - W_{j-1} = \omega(P_j) = |C_j| \qquad (1 \leq j \leq R).$$

Since the P_j are pairwise coprime,

$$h(P_{\leq r}) = \prod_{j=1}^{r} h(P_j).$$

Iterating (11) and using $\left|\mathcal{Q}_j'\right| \geq |\mathcal{Q}_j|/P_j$ for $1 \leq j < r$ therefore gives, for every $1 \leq r \leq R$,

$$|\mathcal{Q}_r| > S' \prod_{j=1}^{r-1} \frac{1}{P_j} \prod_{j=1}^{r} \frac{1}{\Lambda^{w_j} h(P_j)^2} = \frac{S'}{P_{\leq r-1}\, h(P_{\leq r})^2\, \Lambda^{W_r}}. \tag{12}$$

On the other hand, each $q \in \mathcal{Q}_r$ has the form $q = P_{\leq r} m$, where

$$m \leq \frac{x}{P_{\leq r}}, \qquad \omega(m) = K - W_r.$$

Lemma 3.2 therefore gives

$$|\mathcal{Q}_r| \leq \frac{x}{P_{\leq r}} L(-d/2 + o(1), x)\,(\log x)^{W_r/2}, \tag{13}$$

and the $o(1)$ here is uniform in r because $0 \leq W_r \leq K \leq 3M$. Because $P_{\leq r}$ is an exact divisor of every $q \in \mathcal{Q}_r$, Proposition 3.1 gives $h(P_{\leq r}) \leq h(q) \leq e^{\sqrt{X}}$. Comparing (12) and (13) yields (9). □

5. The final optimization

Proof of Theorem 1.1. Let $\mathcal{Q}$ be extremal, $S = f(x)$, and choose $\mathcal{Q}'$ as in Proposition 3.1. Write $S' = |\mathcal{Q}'|$ and define $\sigma = \sigma(x) \geq 0$ by

$$S' = xL(-\sigma, x)\,.$$

The non-negativity follows from $S' \leq x$. Let $K = dM$ and apply Proposition 4.2. Multiplying (9) over $1 \leq r \leq R$ and using (8) gives

$$xL(-2, x) \leq \left(\frac{x}{S'} L(-d/2 + o(1), x)\right)^R (\log x)^{T/2} \Lambda^T e^{2R\sqrt{X}}, \qquad T := \sum_{r=1}^{R} W_r. \tag{14}$$

The $o(1)$ remains harmless after multiplication: by the uniformity in Proposition 4.2 and the bound $R \leq K \leq 3M$, its total logarithmic contribution is $R\, o(1)\, Z = o(X)$.

We need only one elementary bound on T. Since $w_r := W_r - W_{r-1} \geq 1$ and $\sum_{r=1}^{R} w_r = K$,

$$T = \sum_{r=1}^{R} (R - r + 1) w_r \leq R(K - R + 1) + \frac{R(R-1)}{2} = RK - \frac{R^2}{2} + O(R). \tag{15}$$

Write $R = cM$. Then $0 < c \leq d \leq 3$. Also $T = O(M^2)$,

$$\log \Lambda = O(\log\log(eK)) = o(Y), \qquad R\sqrt{X} = O(X/\sqrt{Y}) = o(X),$$

so $\Lambda^T e^{2R\sqrt{X}} = e^{o(X)}$.

Taking logarithms in (14), using (15), and dividing by $X = \log x$, we obtain

$$1 + o(1) \leq c(\sigma - d/2) + \frac{cd}{2} - \frac{c^2}{4} + o(1) = c\sigma - \frac{c^2}{4} + o(1).$$

For $\sigma \geq 0$ and $c > 0$,

$$c\sigma - \frac{c^2}{4} \leq \sigma^2,$$

with equality possible only at $c = 2\sigma$. Hence $\sigma \geq 1 + o(1)$, and so

$$S' \leq xL(-1 + o(1), x)\,.$$

Proposition 3.1 gives $S \leq S' L(o(1), x)$; therefore

$$f(x) = S \leq xL(-1 + o(1), x)\,.$$

The matching lower bound is the left-hand side of (1), proved in [3]. This completes the proof. ☐

Lemma 5.1. *For every fixed $\beta > 0$,*

$$\int_m^\infty \frac{dt}{tL(\beta,t)} = L(-\beta + o(1), m) \qquad (m \to \infty).$$

Proof. Put $u = \log t$, $u_0 = \log m$, and $g(u) = \sqrt{u \log u}$. Then

$$\int_m^\infty \frac{dt}{tL(\beta,t)} = \int_{u_0}^\infty e^{-\beta g(u)}\, du.$$

For the lower bound,

$$\int_{u_0}^{u_0+1} e^{-\beta g(u)}\, du \geq e^{-\beta g(u_0+1)} = e^{-\beta g(u_0) + o(g(u_0))} = L(-\beta + o(1), m)\,.$$

For the upper bound, split the integral into dyadic intervals in u:

$$\int_{u_0}^\infty e^{-\beta g(u)}\, du \leq \sum_{j\geq 0} \int_{2^j u_0}^{2^{j+1}u_0} e^{-\beta g(u)}\, du \leq \sum_{j \geq 0} 2^j u_0\, e^{-\beta g(2^j u_0)}.$$

Since

$$g(2^j u_0) = \sqrt{2^j u_0(\log u_0 + j \log 2)} \geq 2^{j/2} g(u_0),$$

and

$$\sup_{j \geq 0} \frac{\log(2^j u_0)}{2^{j/2} g(u_0)} \to 0 \qquad (m \to \infty),$$

the jth summand is at most $\exp\{-\beta 2^{j/2} g(u_0) + o(g(u_0))\}$. The resulting series is dominated by its first term, so the whole integral is at most $e^{-\beta g(u_0) + o(g(u_0))} = L(-\beta + o(1), m)$. ☐

Proof of Corollary 1.2. Fix $\delta > 0$. By Theorem 1.1, for all sufficiently large t we have

$$f(t) \leq tL(-1+\delta, t)\,.$$

Let $m < n_1 < \cdots < n_k$ be any admissible finite sequence, and put

$$A(t) := \#\{i : n_i \leq t\}.$$

Then $A(t) \leq f(t)$ for all t, and partial summation gives

$$\sum_{i=1}^k \frac{1}{n_i} = \int_m^\infty \frac{A(t)}{t^2}\, dt \leq \int_m^\infty \frac{f(t)}{t^2}\, dt \leq \int_m^\infty \frac{dt}{tL(1-\delta, t)} = L(-1+\delta+o(1), m)$$

for all sufficiently large m, by Lemma 5.1. Since $\delta > 0$ is arbitrary, this gives

$$\epsilon_m \leq L(-1 + o(1), m)\,.$$

For the lower bound, set

$$x := mL(2, m)\,.$$

Then $\log x = \log m + O(\sqrt{\log m \log\log m})$, so $L(1, x) = L(1 + o(1), m)$. By Theorem 1.1 there exist $f(x) = xL(-1 + o(1), x)$ pairwise disjoint congruences with distinct moduli at most x. At most m of these moduli are $\leq m$, so after discarding them we retain at least $f(x) - m$ moduli in $(m, x]$. Therefore

$$\epsilon_m \geq \frac{f(x) - m}{x}.$$

Now

$$\frac{f(x)}{m} = \frac{x}{m} L(-1 + o(1), x) = L(1 + o(1), m) \to \infty,$$

so $m = o(f(x))$, and hence

$$\frac{f(x)-m}{x} = \frac{f(x)}{x}(1+o(1)) = L(-1+o(1),x) = L(-1+o(1),m)\,.$$

Thus $\epsilon_m \geq L(-1+o(1),m)$, completing the proof. □

Remark 5.2. The calculation above also explains the role of the spread input. The BFV chain already produces the main-scale terms $c(\sigma - d/2)$ and $cd/2 - c^2/4$. The spread-core lemma contributes only $T \log \Lambda = o(\log x)$ to the logarithm of the product inequality. Retaining the older K^w loss contributes at the main scale and leads instead to the previously published constant $\sqrt{3}/2$.

References

[1] T. F. Bloom, *Erdős Problem #202*, https://www.erdosproblems.com/202.
[2] T. F. Bloom, *Erdős Problem #1190*, https://www.erdosproblems.com/1190.
[3] R. de la Bretèche, K. Ford and J. Vandehey, *On non-intersecting arithmetic progressions*, Acta Arith. **157** (2013), no. 4, 381–392.
[4] Y.-G. Chen, *On disjoint arithmetic progressions*, Acta Arith. **118** (2005), no. 2, 143–148.
[5] E. S. Croot III, *On non-intersecting arithmetic progressions*, Acta Arith. **110** (2003), no. 3, 233–238.
[6] P. Erdős and E. Szemerédi, *On a problem of P. Erdős and S. Stein*, Acta Arith. **15** (1968), 85–90.
[7] J. Park and H. T. Pham, *A proof of the Kahn–Kalai conjecture*, J. Amer. Math. Soc. **37** (2024), no. 1, 235–243.