ERDŐS'S RATIO-GROWTH PROBLEM AND A QUANTITATIVE HAIGHT-TYPE THEOREM

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Abstract. For a measurable set $S \subset (0, \infty)$, write

$$A_S(x) = \mathrm{Leb}(S \cap (0, x)).$$

Erdős asked how fast $A_S(x)$ can tend to infinity when S has infinite measure and no quotient of two distinct points of S is an integer. We prove the sharp criterion: for every non-decreasing $F\colon [1, \infty) \to [0, \infty)$ with $F(x) \to \infty$, such an S exists with $A_S(x) \geq F(x)$ eventually if and only if

$$\int_1^\infty \frac{F(x)}{x^2}\, dx < \infty.$$

More generally, the same criterion holds for any locally finite forbidden ratio set $\Gamma \subset (1, \infty)$ that contains a full geometric progression. The sufficient direction is stronger: every nonempty locally finite Γ admits all non-decreasing targets satisfying the integral condition. The proof uses logarithmic coordinates, simultaneous recurrence for the finitely many relevant ratios at each scale, and a summable block-deletion argument. We also prove the corresponding sequence theorem: for arbitrary non-negative finite-valued targets, the exact obstruction to achieving $A_S(X_k) \geq F(X_k)$ along some sequence $X_k \to \infty$ is $\liminf_{x\to\infty} F(x)/x = 0$.

1. Introduction

Erdős recorded the following question of W. M. Schmidt [5]; see also [1, Problem 1195]. Is there a measurable set $S \subset \mathbb{R}$ of infinite measure such that no quotient x/y of two distinct points of S is an integer? If so, how fast can $\mathrm{Leb}(S \cap (0, x))$ tend to infinity? Haight and Szemerédi independently proved existence [3, 6]. Erdős observed the elementary consequence $\mathrm{Leb}(S \cap (0, x)) = o(x)$ and asked for sharper growth information [2, p. 101]. Haight later showed that one can avoid any prescribed sequence of ratios with no finite accumulation point in $(0, \infty)$ [4]. This paper gives a sharp quantitative version.

We write

$$A_S(x) \coloneqq \mathrm{Leb}(S \cap (0, x)).$$

Theorem 1.1 (Erdős's integer-ratio growth problem). *Let $F\colon [1, \infty) \to [0, \infty)$ be non-decreasing with $F(x) \to \infty$. There exists a measurable set $S \subset (0, \infty)$ of infinite Lebesgue measure such that x/y is never an integer for distinct $x, y \in S$ and*

$$A_S(x) \geq F(x)$$

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for all sufficiently large x if and only if

$$\int_1^\infty \frac{F(x)}{x^2}\,dx < \infty.$$

Thus $x/(\log x)^{1+\varepsilon}$ is attainable, after changing the target on a bounded interval, whereas $x/\log x$ is not. Erdős used the strict inequality $A_S(x) > F(x)$; applying the theorem to $F+1$ gives the same formulation whenever the integral for F is finite.

The integral condition is forced already by the forbidden ratios $2, 4, 8, \ldots$. Folding dyadic annuli into one fundamental annulus gives

$$\int_S \frac{dt}{t} \le \log 2,$$

and Tonelli's theorem converts this to

$$\int_0^\infty \frac{A_S(x)}{x^2}\,dx < \infty.$$

The main work is to show that the remaining integer ratios impose no larger asymptotic cost.

We now state the general form. It is enough to consider forbidden ratios larger than 1: for an unordered pair of positive distinct points, forbidding a ratio $g < 1$ is the same as forbidding $g^{-1} > 1$.

A set $\Gamma \subset (1,\infty)$ is *locally finite* if $\Gamma \cap [1,T]$ is finite for every finite T. Such a set is at most countable. If it is nonempty, then

$$\lambda_\Gamma := \min \Gamma > 1,$$

because, for any $\gamma_0 \in \Gamma$, the finite nonempty set $\Gamma \cap [1,\gamma_0]$ has a least element. A measurable set $S \subset (0,\infty)$ is Γ-*free* if

$$\frac{x}{y} \notin \Gamma \qquad (x, y \in S).$$

Since $1 \notin \Gamma$, allowing $x = y$ in this definition is harmless; equivalently, $y/x \notin \Gamma$ whenever $x < y$ lie in S.

Theorem 1.2 (Sharp growth for ratio sets containing a geometric progression). *Let $\Gamma \subset (1,\infty)$ be locally finite and suppose that, for some $r > 1$,*

$$\{r^m : m = 1, 2, 3, \ldots\} \subset \Gamma.$$

Let $F\colon [1,\infty) \to [0,\infty)$ be non-decreasing, finite-valued, and satisfy $F(x) \to \infty$. Then there exists a Γ-free measurable set $S \subset (0,\infty)$ of infinite Lebesgue measure such that

$$A_S(x) \ge F(x)$$

for all sufficiently large x if and only if

$$\int_1^\infty \frac{F(x)}{x^2}\,dx < \infty.$$

Theorem 1.1 is the case $\Gamma = \{2, 3, 4, \ldots\}$, which contains $2, 4, 8, \ldots$. The same result applies, for instance, to square ratios $\{n^2 : n \ge 2\}$, which contain $4, 4^2, 4^3, \ldots$.

The lower bound requires no geometric progression.

Theorem 1.3 (Quantitative Haight-type theorem). *Let $\Gamma \subset (1,\infty)$ be nonempty and locally finite. Let $F\colon [1,\infty) \to [0,\infty)$ be non-decreasing, finite-valued, and satisfy $F(x) \to \infty$. If*

$$\int_1^\infty \frac{F(x)}{x^2}\,dx < \infty,$$

then there exists a Γ-free measurable set $S \subset (0,\infty)$ of infinite Lebesgue measure such that

$$A_S(x) \geq F(x)$$

for all sufficiently large x.

The proof works in logarithmic coordinates, where multiplication by $\gamma \in \Gamma$ becomes translation by $\log\gamma$. In a new annulus we place a rapidly oscillating logarithmic comb. The only ratios that can connect this new block to the earlier construction form a finite set. A simultaneous recurrence makes those finitely many translations almost preserve the comb, so the old points whose forbidden multiples hit the new block occupy only a small proportion of each old block. A summable deletion removes all future conflicts while preserving the prescribed annular masses.

The geometric-progression hypothesis in Theorem 1.2 is used only for the converse. It cannot be replaced by local finiteness alone: if $\Gamma = \{2\}$, then

$$\bigcup_{k\geq 0} [4^k, 2\cdot 4^k)$$

is $\{2\}$-free and satisfies

$$\liminf_{x\to\infty} \frac{\mathrm{Leb}(S\cap(0,x))}{x} = \frac{1}{3}.$$

We also prove the sequence analogue.

Theorem 1.4 (Growth along a sequence). *Let $\Gamma \subset (1,\infty)$ be locally finite and suppose that $\{r^m : m \geq 1\} \subset \Gamma$ for some $r > 1$. Let $F\colon [1,\infty) \to [0,\infty)$ be finite-valued; no monotonicity is assumed. There exists a Γ-free measurable set $S \subset (0,\infty)$ of infinite Lebesgue measure and a sequence $X_k \to \infty$ such that*

$$A_S(X_k) \geq F(X_k) \qquad (k \geq 1)$$

if and only if

$$\liminf_{x\to\infty} \frac{F(x)}{x} = 0.$$

The sufficient implication remains true for every nonempty locally finite $\Gamma \subset (1,\infty)$.

Thus eventual growth and growth along a sequence have different answers, as Erdős anticipated.

2. The upper bound from a geometric progression

Proposition 2.1 (Logarithmic packing bound). *Let $r > 1$ and let $S \subset (0, \infty)$ be measurable. Suppose that no quotient of two distinct points of S belongs to $\{r^m : m \geq 1\}$. Then*

$$\int_S \frac{dt}{t} \leq \log r.$$

Consequently,

$$\int_0^\infty \frac{A_S(x)}{x^2}\, dx \leq \log r.$$

Proof. For $k \in \mathbb{Z}$, set

$$B_k := r^{-k}\Big(S \cap [r^k, r^{k+1})\Big) \subset [1, r).$$

The sets B_k are pairwise disjoint: if $u \in B_i \cap B_j$ with $i < j$, then $r^i u, r^j u \in S$ and their quotient is r^{j-i}. Since dt/t is dilation-invariant,

$$\int_S \frac{dt}{t} = \sum_{k \in \mathbb{Z}} \int_{B_k} \frac{du}{u} \leq \int_1^r \frac{du}{u} = \log r.$$

Tonelli's theorem gives

$$\int_0^\infty \frac{A_S(x)}{x^2}\, dx = \int_S \int_t^\infty \frac{dx}{x^2}\, dt = \int_S \frac{dt}{t},$$

and the result follows. □

Corollary 2.2. *Let $r > 1$, and let S be as in Proposition 2.1. Then $A_S(x) = o(x)$. Moreover, if $X_0 > 0$ and $F\colon [X_0, \infty) \to [0, \infty)$ is measurable with*

$$A_S(x) \geq F(x) \qquad (x \geq X_0),$$

then

$$\int_{X_0}^\infty \frac{F(x)}{x^2}\, dx < \infty.$$

Proof. The integral assertion is immediate. To prove $A_S(x) = o(x)$, fix $R > 0$. For $x > R$,

$$A_S(x) \leq A_S(R) + \mathrm{Leb}(S \cap (R, x)) \leq A_S(R) + x \int_{S \cap (R,x)} \frac{dt}{t}.$$

Thus

$$\limsup_{x \to \infty} \frac{A_S(x)}{x} \leq \int_{S \cap (R, \infty)} \frac{dt}{t}.$$

The right side tends to 0 as $R \to \infty$, by Proposition 2.1. □

3. Logarithmic block placement

Fix $b > 1$. Write

$$\|u\|_{\mathbb{Z}} := \operatorname{dist}(u, \mathbb{Z}),$$

and, for $0 < \alpha < 1/2$ and $q \in \mathbb{N}$, define the logarithmic comb

$$L_{q,b}(\alpha) := \{x > 0 : \|q \log_b x\|_{\mathbb{Z}} < \alpha\}.$$

Lemma 3.1 (Periodic averaging). *Let $g \in L^1(a, b_0)$ and let p be a bounded Riemann-integrable 1-periodic function. Then*

$$\lim_{q\to\infty} \int_a^{b_0} g(u)p(qu)\, du = \left(\int_0^1 p(v)\, dv\right) \int_a^{b_0} g(u)\, du.$$

Consequently, for every measurable $P \subset [c, d] \subset (0, \infty)$ and every $0 < \alpha < 1/2$,

$$\operatorname{Leb}(P \cap L_{q,b}(\alpha)) \longrightarrow 2\alpha \operatorname{Leb}(P) \qquad (q \to \infty).$$

Proof. For $g = \mathbf{1}_{[s,t]}$,

$$\int_s^t p(qu)\, du = \frac{1}{q} \int_{qs}^{qt} p(v)\, dv \longrightarrow (t - s) \int_0^1 p(v)\, dv.$$

Linearity gives the result for step functions, and L^1 approximation gives it for general g, since the error is bounded by $\|p\|_\infty \|g - h\|_1$.

For the consequence, put $u = \log_b x$. On $[c, d]$ this is a bi-Lipschitz C^1 change of variables, so measurability is preserved and $dx = (\log b)b^u\, du$. Apply the first part to

$$g(u) = (\log b)b^u \mathbf{1}_{\log_b P}(u), \qquad p(v) = \mathbf{1}_{\{\|v\|_{\mathbb{Z}} < \alpha\}}.$$

The function p is Riemann integrable, 1-periodic, and has mean 2α. □

Lemma 3.2 (Large simultaneous returns). *Let $\theta_1, \ldots, \theta_m \in \mathbb{R}$ and let $\eta > 0$. For every $Q > 0$ there is an integer $q > Q$ such that*

$$\|q\theta_j\|_{\mathbb{Z}} < \eta \qquad (1 \le j \le m).$$

Proof. Let $\theta = (\theta_1, \ldots, \theta_m) \in \mathbb{T}^m$ and $K = \overline{\{n\theta : n \in \mathbb{Z}\}}$. Let $U \subset K$ be the neighbourhood of 0 on which all coordinates have distance $< \eta$ from 0 modulo 1. Finitely many translates $n_1\theta + U, \ldots, n_M\theta + U$ cover K. Choose $N > Q + \max_j |n_j|$. Since $N\theta \in K$, some $N\theta - n_j\theta$ lies in U; then $q = N - n_j > Q$ has the required property. □

For $H \subset (0, \infty)$ define its Γ-shadow by

$$\Phi_\Gamma(H) := \{x > 0 : \gamma x \in H \text{ for some } \gamma \in \Gamma\}.$$

Lemma 3.3 (Block-shadow lemma). *Let $\Gamma \subset (1, \infty)$ be nonempty and locally finite, and choose $1 < b < \lambda_\Gamma$. There are constants $c_b, C_b > 0$, depending only on b, with the following property. Let $N \in \mathbb{Z}$, and let $P_1, \ldots, P_\ell$ be a finite family, possibly empty, of measurable subsets of $(0, b^N)$. Assume that $\bigcup_i P_i$ is bounded away from 0 whenever it is nonempty. For every $0 < \rho < 1/4$ there is an open set $H \subset (b^N, b^{N+1})$ such that*

$$\operatorname{Leb}(H) \ge c_b \rho b^N$$

and

$$\mathrm{Leb}(P_i \cap \Phi_\Gamma(H)) \le C_b\rho\,\mathrm{Leb}(P_i) \qquad (1 \le i \le \ell).$$

Proof. Let $P = \bigcup_i P_i$. If $P \neq \varnothing$, choose $\delta > 0$ with $P \subset [\delta, b^N)$, and set

$$\mathcal{G} := \{\gamma \in \Gamma : \gamma P \cap (b^N, b^{N+1}) \neq \varnothing\}.$$

If $\gamma \in \mathcal{G}$, then $\gamma x < b^{N+1}$ for some $x \in P \subset [\delta, b^N)$; hence $\gamma < b^{N+1}/\delta$. Thus $\mathcal{G} \subset \Gamma \cap [1, b^{N+1}/\delta]$, so $\mathcal{G}$ is finite. If $P = \varnothing$, put $\mathcal{G} = \varnothing$.

Set

$$\alpha = \rho/16, \qquad \beta = \rho/8, \qquad \eta = \rho/16.$$

By Lemma 3.1, all sufficiently large integers q satisfy

$$\mathrm{Leb}\Big((b^N, b^{N+1}) \cap L_{q,b}(\alpha)\Big) \ge \alpha(b-1)b^N, \tag{1}$$

$$\mathrm{Leb}(P_i \cap L_{q,b}(\beta)) \le 4\beta\,\mathrm{Leb}(P_i) \qquad (1 \le i \le \ell). \tag{2}$$

For null P_i the second estimate is automatic; otherwise it follows from convergence to $2\beta\,\mathrm{Leb}(P_i)$.

Choose such a large q. If $\mathcal{G} \neq \varnothing$, choose it also, by Lemma 3.2, so that

$$\|q \log_b \gamma\|_{\mathbb{Z}} < \eta \qquad (\gamma \in \mathcal{G}). \tag{3}$$

This is possible because $\mathcal{G}$ is finite. Now put

$$H = (b^N, b^{N+1}) \cap L_{q,b}(\alpha).$$

Then H is open and, by (1),

$$\mathrm{Leb}(H) \ge \frac{b-1}{16}\rho b^N.$$

If $x \in P_i \cap \Phi_\Gamma(H)$, then $\gamma x \in H$ for some $\gamma \in \Gamma$, so necessarily $\gamma \in \mathcal{G}$. Hence (3) and $\gamma x \in L_{q,b}(\alpha)$ give

$$\|q \log_b x\|_{\mathbb{Z}} < \alpha + \eta = \beta.$$

Therefore $P_i \cap \Phi_\Gamma(H) \subset P_i \cap L_{q,b}(\beta)$, and (2) gives

$$\mathrm{Leb}(P_i \cap \Phi_\Gamma(H)) \le 4\beta\,\mathrm{Leb}(P_i) = \frac{\rho}{2}\,\mathrm{Leb}(P_i).$$

The lemma follows with $c_b = (b-1)/16$ and $C_b = 1$. $\square$

4. The lower construction

The next lemma is the construction used in both lower-bound arguments. The base b is chosen below the smallest forbidden ratio, so a single b-adic annulus contains no forbidden quotient. Future blocks may create conflicts with old blocks; those old points are deleted, and the total deletion is summable.

Lemma 4.1 (Annular realization). *Let $\Gamma \subset (1, \infty)$ be nonempty and locally finite, choose $1 < b < \lambda_\Gamma$, and let c_b, C_b be as in Lemma 3.3. Choose $K > 0$ with $c_b K \ge 2$. Let $J \in \mathbb{Z}$, and let $(m_j)_{j \ge J}$ be nonnegative numbers such that, with*

$$\rho_j := K\frac{m_j}{b^j},$$

one has

$$\rho_j < \frac{1}{4} \qquad (j \geq J), \qquad C_b \sum_{j>i} \rho_j \leq \frac{1}{2} \qquad (i \geq J).$$

Then there are measurable sets $S_j \subset (b^j, b^{j+1})$ *with* $\operatorname{Leb}(S_j) \geq m_j$ *such that* $S = \bigcup_{j \geq J} S_j$ *is* Γ*-free. If* $\sum_{j \geq J} m_j = \infty$*, then* $\operatorname{Leb}(S) = \infty$.

Proof. Construct open sets $H_j \subset (b^j, b^{j+1})$ inductively. If $m_j = 0$, take $H_j = \varnothing$. If $m_j > 0$, apply Lemma 3.3 with $N = j$, parameter ρ_j, and the previous nonempty blocks H_i for $J \leq i < j$, as the family P_i. This family is finite, and its union is contained in $[b^J, b^j)$, hence is bounded away from 0. We get

$$\operatorname{Leb}(H_j) \geq c_b \rho_j b^j = c_b K m_j \geq 2m_j \tag{4}$$

and, for every $i < j$,

$$\operatorname{Leb}(H_i \cap \Phi_\Gamma(H_j)) \leq C_b \rho_j \operatorname{Leb}(H_i), \tag{5}$$

trivially also when $H_i = \varnothing$.

After all H_j are chosen, define

$$S_i := H_i \setminus \bigcup_{j>i} \Big(H_i \cap \Phi_\Gamma(H_j) \Big). \tag{6}$$

The set $\Phi_\Gamma(H_j) = \bigcup_{\gamma \in \Gamma} \gamma^{-1} H_j$ is open, since Γ is countable, so each S_i is measurable. Also

$$\begin{aligned} \operatorname{Leb}(S_i) &\geq \operatorname{Leb}(H_i) - \sum_{j>i} \operatorname{Leb}(H_i \cap \Phi_\Gamma(H_j)) \\ &\geq \operatorname{Leb}(H_i) \left(1 - C_b \sum_{j>i} \rho_j \right) \geq \tfrac{1}{2} \operatorname{Leb}(H_i) \geq m_i. \end{aligned}$$

Let $x < y$ be points of S. If they lie in the same annulus (b^i, b^{i+1}), then $1 < y/x < b < \lambda_\Gamma$, so $y/x \notin \Gamma$. If $x \in S_i$ and $y \in S_j$ with $i < j$, then $y/x \in \Gamma$ would imply $x \in H_i \cap \Phi_\Gamma(H_j)$, contrary to (6). Thus S is Γ-free. The final assertion follows from the disjointness of the annuli. $\square$

Proof of Theorem 1.3. Choose $1 < b < \lambda_\Gamma$. Since F is non-decreasing and

$$\int_1^\infty \frac{F(x)}{x^2}\, dx < \infty,$$

we have

$$\sum_{j=1}^{\infty} \frac{F(b^j)}{b^j} < \infty, \tag{7}$$

because

$$\int_{b^j}^{b^{j+1}} \frac{F(x)}{x^2}\, dx \geq \frac{b-1}{b} \frac{F(b^j)}{b^j}.$$

Let c_b, C_b be as in Lemma 3.3, and choose $K > 0$ with $c_b K \geq 2$.

For $J \geq 1$, define

$$U_J := \frac{F(b^{J+2})}{b^J} + \sum_{j>J} \frac{F(b^{j+2}) - F(b^{j+1})}{b^j}.$$

Then

$$0 \leq U_J \leq b^2 \sum_{\ell \geq J+2} \frac{F(b^\ell)}{b^\ell},$$

so $U_J \to 0$ by (7). Choose J so large that

$$KU_J < \frac{1}{4}, \qquad C_b K U_J \leq \frac{1}{2}.$$

Set

$$m_J := F(b^{J+2}), \qquad m_j := F(b^{j+2}) - F(b^{j+1}) \quad (j > J).$$

Then $m_j \geq 0$,

$$\sum_{j=J}^{M} m_j = F(b^{M+2}) \qquad (M \geq J),$$

and $\sum_{j \geq J} m_j / b^j = U_J$. Since $F(x) \to \infty$, also $\sum_{j \geq J} m_j = \infty$. Thus the hypotheses of Lemma 4.1 hold with $\rho_j = K m_j / b^j$, and we obtain a Γ-free set $S = \bigcup_{j \geq J} S_j$ of infinite measure with $S_j \subset (b^j, b^{j+1})$ and $\mathrm{Leb}(S_j) \geq m_j$.

If $N \geq J+1$ and $b^N \leq x < b^{N+1}$, then

$$A_S(x) \geq \sum_{j=J}^{N-1} \mathrm{Leb}(S_j) \geq \sum_{j=J}^{N-1} m_j = F(b^{N+1}) \geq F(x).$$

Hence $A_S(x) \geq F(x)$ for all $x \geq b^{J+1}$. □

5. Proofs of the sharp and sequence theorems

Proof of Theorem 1.2. If S exists, then it avoids the geometric progression $\{r^m : m \geq 1\}$. Corollary 2.2 therefore gives the integral condition, since a finite-valued monotone function is measurable. Conversely, the integral condition gives the desired Γ-free set by Theorem 1.3. □

Proof of Theorem 1.1. Apply Theorem 1.2 with $\Gamma = \{2, 3, 4, \dots\}$ and $r = 2$. □

Proof of Theorem 1.4. For necessity, Corollary 2.2 gives $A_S(x) = o(x)$; hence

$$0 \leq \frac{F(X_k)}{X_k} \leq \frac{A_S(X_k)}{X_k} \to 0.$$

Assume conversely that $\liminf_{x \to \infty} F(x)/x = 0$. Choose $1 < b < \lambda_\Gamma$. The geometric progression is not used in this direction. Pick $X_k \to \infty$ so that

$$\frac{F(X_k)}{X_k} < b^{-2} 2^{-k-4}, \qquad \frac{k}{X_k} < b^{-2} 2^{-k-4},$$

and so that

$$n_k := \lfloor \log_b X_k \rfloor - 1$$

is strictly increasing. Then $b^{n_k+1} \leq X_k < b^{n_k+2}$ and $b^{n_k} > X_k / b^2$.

Put $M_k = \max\{F(X_k), k\}$. Since

$$\frac{M_k}{X_k} \le \frac{F(X_k)}{X_k} + \frac{k}{X_k} < b^{-2}2^{-k-3},$$

we have

$$\sum_k \frac{M_k}{b^{n_k}} \le b^2 \sum_k \frac{M_k}{X_k} < \infty, \qquad \sum_k M_k = \infty.$$

Let c_b, C_b be as in Lemma 3.3, and choose $K > 0$ with $c_b K \ge 2$. Choose k_0 so large that

$$K \sum_{k \ge k_0} \frac{M_k}{b^{n_k}} < \frac{1}{4}, \qquad C_b K \sum_{k \ge k_0} \frac{M_k}{b^{n_k}} \le \frac{1}{2}.$$

Set $J = n_{k_0}$ and define, for $j \ge J$,

$$m_{n_k} := M_k \quad (k \ge k_0), \qquad m_j := 0 \quad (j \notin \{n_k : k \ge k_0\}).$$

Lemma 4.1 gives a Γ-free set S of infinite measure such that

$$\mathrm{Leb}\Big(S \cap (b^{n_k}, b^{n_k+1})\Big) \ge M_k \qquad (k \ge k_0).$$

Because $(b^{n_k}, b^{n_k+1}) \subset (0, X_k)$,

$$A_S(X_k) \ge M_k \ge F(X_k) \qquad (k \ge k_0).$$

Relabeling this tail of the sequence completes the proof. □

6. Remarks and examples

Remark 6.1 (Other forbidden ratio sets). The same sharp criterion applies to any locally finite forbidden ratio set containing a full geometric progression. Thus Theorem 1.2 applies to square ratios $\Gamma = \{n^2 : n \ge 2\}$, to perfect k-th power ratios, and to any larger locally finite set containing $\{r^m : m \ge 1\}$.

Remark 6.2 (Why a geometric progression is an upper-bound hypothesis). Theorem 1.3 works for every nonempty locally finite $\Gamma \subset (1, \infty)$, but Theorem 1.2 needs an additional upper-bound hypothesis. For example, with $\Gamma = \{2\}$,

$$S = \bigcup_{k \ge 0} [4^k, 2 \cdot 4^k)$$

is $\{2\}$-free: if $x \in [4^k, 2 \cdot 4^k)$, then $2x \in [2 \cdot 4^k, 4 \cdot 4^k)$, outside S. Moreover

$$\liminf_{x \to \infty} \frac{A_S(x)}{x} = \frac{1}{3}.$$

So local finiteness alone cannot imply the integral obstruction.

Remark 6.3 (Non-monotone targets). Since A_S is non-decreasing, a locally bounded nonnegative target F is controlled by its running envelope

$$F^*(x) := \sup_{X_0 \le t \le x} F(t).$$

For ratio sets covered by Theorem 1.2, eventual domination of F is possible exactly when, for some X_0, this envelope is finite-valued and

$$\int_{X_0}^{\infty} \frac{F^*(x)}{x^2}\,dx < \infty.$$

Necessity follows because $A_S(t) \geq F(t)$ for all $t \geq X_0$ implies $A_S(x) \geq F^*(x)$ for all $x \geq X_0$. For sufficiency, apply Theorem 1.2 to any non-decreasing finite-valued function tending to infinity that majorizes F^*, for example $F^*(x) + \log\log(e+x)$ after extending harmlessly to $[1,\infty)$. If F^* is not finite-valued, no finite-valued counting function A_S can dominate F eventually.

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