Two non-attacking queens on an $n \times n$ board

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How many ways are there to place two non-attacking queens on an $n \times n$ chessboard? That is, the two queens cannot lie on the same row, column, or diagonal as each other. This question was first answered by Édouard Lucas in his 1891 book *Théorie des nombres*.

We first count the number of ways to place two queens on the board, with no restrictions. Since there are n^2 choices for the first queen and $n^2 - 1$ for the second queen, we see that there are

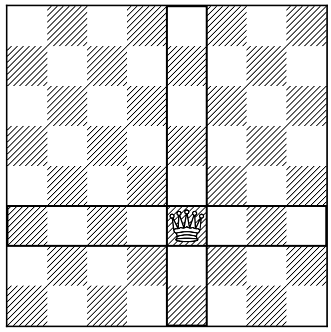
$$\binom{n^2}{2} = \frac{n^2(n^2-1)}{2} \text{ ways}$$

(where we divide by two, since every pair gets counted twice).

Next, we must subtract the number of ways to place two *attacking* queens on the board. The number of ways such that both queens are on the same row or same column is equal to

$$n^2(n-1),$$

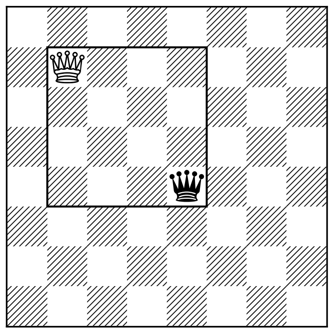
since there are n^2 choices for the first queen, and $2(n-1)$ choices for the second queen; we then divide by two as before.



The queen attacks $n-1$ squares on her row, and $n-1$ squares on her column.

Finally, we must subtract the number of ways two queens can attack each other diagonally. The trick is to notice that, if two queens lie on the same diagonal, then they form opposite corners of a unique $k \times k$ square on the board. There are $(n-k+1)^2$ ways to choose a $k \times k$ square on the board (for example, on an 8×8 board, there are only four 7×7 squares). We can sum up the total number of $k \times k$ squares for each k, starting with how only one $n \times n$ square fits in the board, and how four $(n-1) \times (n-1)$ squares fit, and so on, all the way to the 2×2 square:

$$1^2 + 2^2 + 3^2 + \cdots + (n-1)^2 = \frac{n(n-1)(2n-1)}{6}.$$



Two queens on the same diagonal uniquely define a square.

Each square admits two ways of having diagonally opposite queens; therefore, there are

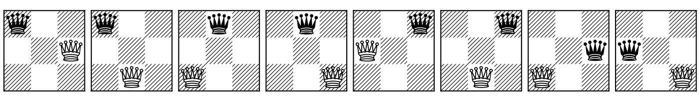
$$\frac{n(n-1)(2n-1)}{6} \times 2 = \frac{n(n-1)(2n-1)}{3}$$

ways for two queens to attack each other diagonally.

Putting everything together, we see that there are

$$\begin{aligned} &\frac{n^2(n^2-1)}{2} - n^2(n-1) - \frac{n(n-1)(2n-1)}{3} \\ &= \frac{1}{6}n(n-1)(n-2)(3n-1) \text{ ways} \end{aligned}$$

to place two non-attacking queens on an $n \times n$ chessboard. □



The eight ways to place two non-attacking queens on a 3×3 board.